MECHANICS

Lecture notes for Phys 111

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Abstract

These notes are intended as an addition to the lectures given in class. They are NOT designed to replace the actual lectures. Some of the notes will contain less information then in the actual lecture, and some will have extra info. Not all formulas which will be needed for exams are contained in these notes. Also, these notes will NOT contain any up to date organizational or administrative information (changes in schedule, assignments, etc.) but only physics. If you notice any typos - let me know at vitaly@njit.edu. For convenience, I will keep all notes in a single file - each time you can print out only the added part. Make sure the file is indeed updated, there is a date indicating the latest modification. There is also a Table of Contents, which is automatically updated. For convenience, the file with notes will be both in postscript and pdf formats. A few other things:

Graphics: Some of the graphics is deliberately unfinished, so that we have what to do in class.

Advanced topics: these will not be represented on the exams. Read them only if you are really interested in the material.

Computer: Mostly, the use of a computer will not be required in the lecture part of this course. If I need it (e.g., for graphics), I will use *Mathematica*. You do not have to know this program, but if you are interested I will be glad to explain how it works.

Contents

I. INTRODUCTION

A. Physics and other sciences

in class

B. Point mass

The art physics is the art of idealization. One of the central concepts in mechanics is a

"particle" or "point mass"

i.e. a body the size or structure of which are irrelevant in a given problem. Examples: electron, planet, etc.

C. Units

1. *Standard units*

In SI system the **basic** units are:

$$\boxed{\text{m (meter), kg (kilogram) and s (second)}}$$

Everything else in mechanics is derived. Examples of derived units (may or may not have a special name):
m/s, m/s^2 (no name), $kg \cdot m/s^2$ (Newton), $kg \cdot m^2/s^2$ (Joule), etc.

2. *Conversion of units*

Standard path: all units are converted to SI. E.g., length:

$$1\,in = 0.0254\,m\ ,\quad 1\,ft = 0.3048\,m\ ,\quad 1\,mi \simeq 1609\,m$$

Examples:

$$70\,\frac{mi}{h} = 70\frac{1609\,m}{3600\,s} \simeq 31.3\,\frac{m}{s}$$

$$3\,cm^2 = 3\,\left(10^{-2}\,m\right)^2 = 3\,10^{-4}\,m^2$$

D. Units and dimensional analysis

Verification of units is useful to check the math. More interesting, however, is the possibility to get some insight into a new problem *before* math is done, or before it is even possible. E.g., suppose we do not know the formula for displacement in accelerated motion with $v_0 = 0$. Let us guess, having at our disposal only the mass $m, [kg]$, acceleration $a, [m/s^2]$ and time $t, [s]$. Since all functions exp, sin, etc. can have only a dimensionless argument,

$$\sin t \;\; - \;\; wrong!$$

$$\sin\frac{t}{T} \;\; - \;\; correct$$

Look for a power law

$$x \sim a^\alpha t^\beta m^\gamma \; or \; [m] = [m/s^2]^\alpha [s]^\beta [kg]^\gamma$$

To get correct dimensions,

$$\alpha = 1\,, \quad \beta = 2\,, \quad \gamma = 0$$

(cannot get the coefficient, which is 1/2 but otherwise ok). Important: not too many variables, otherwise could have multiple solutions, which is as good as none: e.g., if $v_0 \neq 0$ is present can construct a dimensionless at/v_0 and any function can be expected.

Advanced. Less trivial example: gravitational waves. What is the speed? Can depend on $g, [m/s^2]$ on $\lambda, [m]$ and on $\rho, [kg/m^3]$

$$v \sim g^\alpha \lambda^\beta \rho^\gamma \; or \; [m/s] = [m/s^2]^\alpha [m]^\beta [kg/m^3]^\gamma$$

From dimensions,

$$\alpha = \beta = 1/2\,, \gamma = 0(!)$$

$$v \sim \sqrt{g\lambda}$$

What is neglected? Depth of the ocean, H. Thus,

$$v_{\max} \sim \sqrt{gH} \sim \sqrt{10 \cdot 4 \cdot 10^3} \sim 200\, m/s$$

(the longest and fastest gravitational wave is tsunami). Note that we know very little about the precise physics, and especially the precise math of the wave, but from dimensional analysis could get a reasonable estimation.

Problems.

Galileo discovered that the period of small oscillations of a pendulum is independent of its amplitude. Use this to find the dependence of the period T on the length of the pendulum l, gravitational acceleration g and, possibly, mass m. Namely, look for

$$T \sim l^{\alpha} g^{\beta} m^{\gamma}$$

and find α, β and γ.

The force created by a string stretched by x meters is given by $F = -kx$ ("Hook's law") where the spring constant k is measured in N/m and $N = kg \cdot m/s^2$. Find the dependence of the period of oscillations T of a body of mass m attached to this spring on the values of k, m and the amplitude A.

II. VECTORS

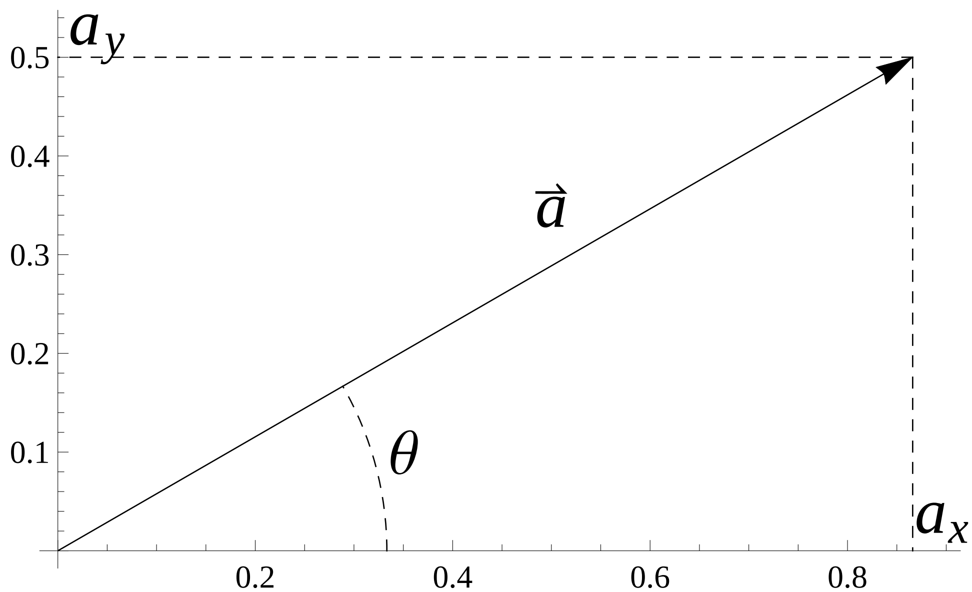
A *vector* is characterized by the following *three* properties:

- has a magnitude
- has direction (Equivalently, has several components in a selected system of coordinates).
- obeys certain addition rules ("rule of parallelogram"). (Equivalently, components of a vector are transformed according to certain rules if the system of coordinates is rotated).

This is in contrast to a *scalar*, which has only magnitude and which is *not* changed when a system of coordinates is rotated.

How do we know which physical quantity is a vector, which is a scalar and which is neither? From experiment (of course). Examples of scalars are mass, time, kinetic energy. Examples of vectors are the displacement, velocity and force.

1. Single vector



Consider a vector $\vec{a}$ with components a_x and a_y (let's talk 2D for a while). There is an associated scalar, namely the magnitude (or length) given by the Pythagorean theorem

$$a \equiv |\vec{a}| = \sqrt{a_x^2 + a_y^2} \tag{1}$$

Note that for a different system of coordinates with axes x', y' the components $a_{x'}$ and $a_{y'}$ can be very different, but the length in eq. (1) , obviously, will not change, which just means that it is a scalar.

Primary example: position vector (note two equivalent forms of notation)

$$\vec{r} = (x, y) = x\vec{i} + y\vec{j}$$

with $|\vec{i}| = |\vec{j}| = 1$.

Polar coordinates:

$$r = \sqrt{x^2 + y^2}\ , \quad \theta = \arctan(y/x)$$

$$x = r\cos\theta\ , \quad y = r\sin\theta$$

Note: arctan might require adding 180^o - always check with a picture!

Another operation allowed on a single vector is multiplication by a scalar. Note that the physical dimension ("units") of the resulting vector can be different from the original, as in $\vec{F} = m\vec{a}$.

2. Two vectors: addition

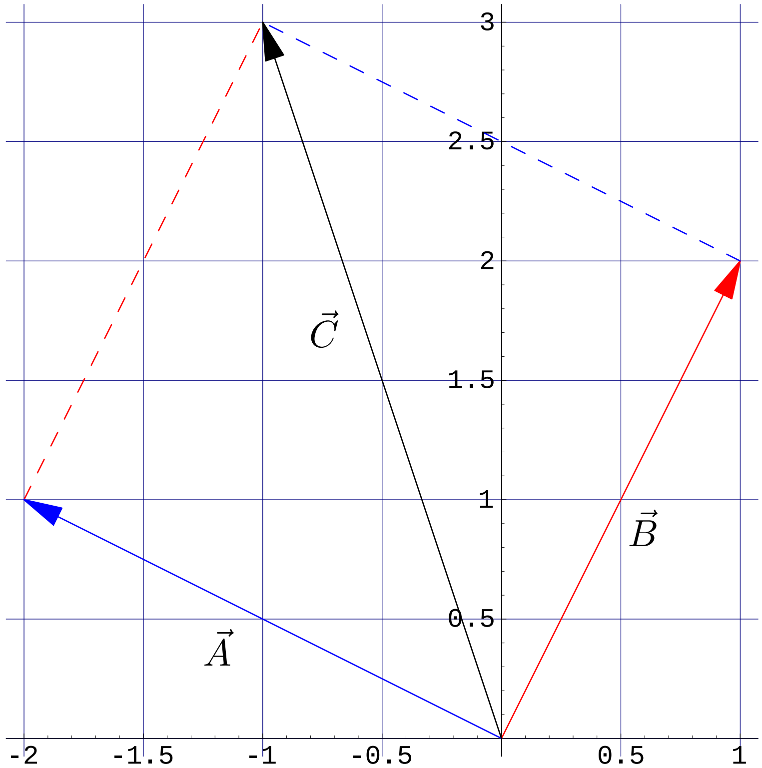


FIG. 1: Adding two vectors: $\vec{C} = \vec{A} + \vec{B}$. Note the use of rule of parallelogram (equivalently, tail-to-head addition rule). Alternatively, vectors can be added by components: $\vec{A} = (-2, 1)$, $\vec{B} = (1, 2)$ and $\vec{C} = (-2 + 1,\ 1 + 2) = (-1, 3)$.

For two vectors, $\vec{a}$ and $\vec{b}$ one can define their sum $\vec{c} = \vec{a} + \vec{b}$ with components

$$c_x = a_x + b_x \ , \quad c_y = a_y + b_y \tag{2}$$

The magnitude of $\vec{c}$ then follows from eq. (1). Note that physical dimensions of $\vec{a}$ and $\vec{b}$ must be identical.

Note: for most problems (except rotation!) it is allowed to carry a vector parallel to itself. Thus, we usually assume that every vector starts at the origin, $(0, 0)$.

3. Two vectors: dot product

If $\vec{a}$ and $\vec{b}$ make an angle ϕ with each other, their scalar (dot) product is defined as $\vec{a} \cdot \vec{b} = ab\cos(\phi)$, or in components

$$\vec{a} \cdot \vec{b} = a_x b_x + a_y b_y \tag{3}$$

A different system of coordinates can be used, with different individual components but with the same result. For two orthogonal vectors $\vec{a} \cdot \vec{b} = 0$. *Preview.* The main application of the scalar product is the concept of work $\Delta W = \vec{F} \cdot \Delta\vec{r}$, with $\Delta\vec{r}$ being the displacement. Force which is perpendicular to displacement does not work!

Example Find angle between 2 vectors.

$$\cos\theta = \frac{\vec{a} \cdot \vec{b}}{ab}$$

Example: Prove the Pythagorean theorem $C^2 = A^2 + B^2$. Proof. Let $\vec{A}$, $\vec{B}$ represent the legs of a right triangle and $\vec{C}$ the hypotenuse - see Fig. 2.

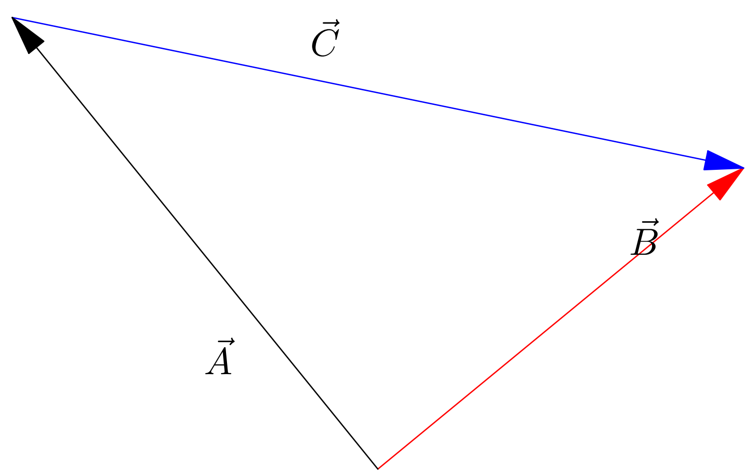


FIG. 2:

One has

$$\vec{C} = \vec{B} - \vec{A}$$

or

$$C^2 = B^2 + A^2 - 2\vec{B} \cdot \vec{A} = A^2 + B^2$$

since $\vec{B} \cdot \vec{A} = 0$ for perpendicular vectors.

Example. The cosine theorem. The same thing, only now $\vec{B} \cdot \vec{A} \neq 0$:

$$C^2 = B^2 + A^2 - 2\vec{B} \cdot \vec{A} = A^2 + B^2 - 2AB\cos\theta$$

θ being the angle between $\vec{A}$ and $\vec{B}$.

4. Two vectors: vector product

At this point we must proceed to the 3D space. Important here is the correct system of coordinates, as in Fig. 3. You can rotate the system of coordinates any way you like, but you cannot reflect it in a mirror (which would switch right and left hands). If $\vec{a}$ and $\vec{b}$ make

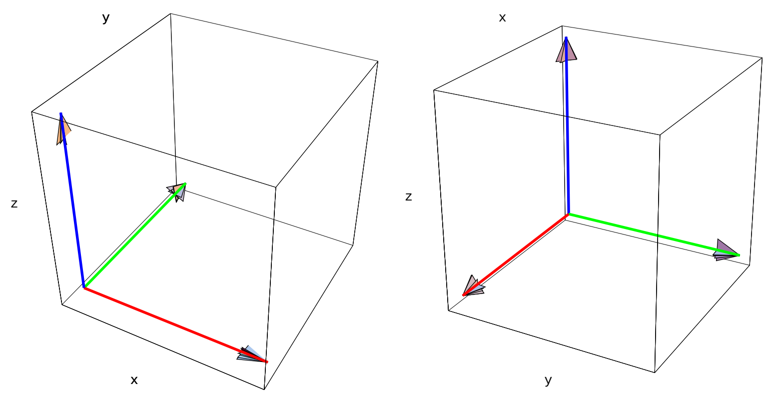


FIG. 3: The correct, "right-hand" systems of coordinates. Checkpoint - curl fingers of the RIGHT hand from x (red) to y (green), then the thumb should point into the z direction (blue). (Note that axes labeling of the figures is outside of the boxes, not necessarily near the corresponding axis; also, for the figure on the right the origin of coordinates is at the *far* end of the box, if it is hard to see in your printout).

an angle $\phi \leq 180^o$ with each other, their vector (cross) product $\vec{c} = \vec{a} \times \vec{b}$ has a magnitude $c = ab\sin(\phi)$. The direction is defined as perpendicular to both $\vec{a}$ and $\vec{b}$ using the following rule: curl the fingers of the right hand from $\vec{a}$ to $\vec{b}$ in the shortest direction (i.e., the angle must be smaller than 180^o). Then the thumb points in the $\vec{c}$ direction. Check with Fig. 4.

Changing the order changes the sign, $\vec{b} \times \vec{a} = -\vec{a} \times \vec{b}$. In particular, $\vec{a} \times \vec{a} = \vec{0}$. More generally, the cross product is zero for any two parallel vectors.

Suppose now a system of coordinates is introduced with unit vectors $\hat{i}$, $\hat{j}$ and $\hat{k}$ pointing in the x, y and z directions, respectively. First of all, if $\hat{i}$, $\hat{j}, \hat{k}$ are written "in a ring", the

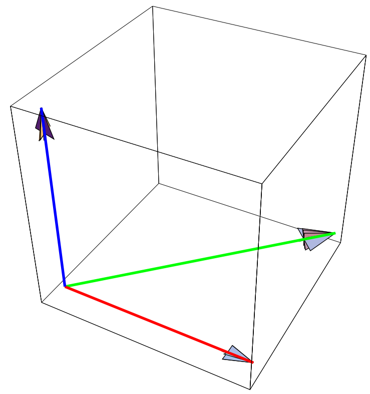


FIG. 4: Example of a cross product $\vec{c}$ (blue) = $\vec{a}$ (red) × $\vec{b}$ (green). (If you have no colors, $\vec{c}$ is vertical in the example, $\vec{a}$ is along the front edge to lower right, $\vec{b}$ is diagonal).

cross product of any two of them equals the third one in clockwise direction, i.e. $\hat{\imath} \times \hat{\jmath} = \hat{k}$, $\hat{\jmath} \times \hat{k} = \hat{\imath}$, etc. (check this for Fig. 3 !). More generally, the cross product is now expressed as a 3-by-3 determinant

$$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix} = \hat{\imath} \begin{vmatrix} a_y & a_z \\ b_y & b_z \end{vmatrix} - \hat{\jmath} \begin{vmatrix} a_x & a_z \\ b_x & b_z \end{vmatrix} + \hat{k} \begin{vmatrix} a_x & a_y \\ b_x & b_y \end{vmatrix} \tag{4}$$

The two-by-two determinants can be easily expanded. In practice, there will be many zeroes, so calculations are not too hard.

Preview. Vector product is most relevant to rotation.

III. 1-DIMENSIONAL MOTION

A. $v = const$

See fig. 5

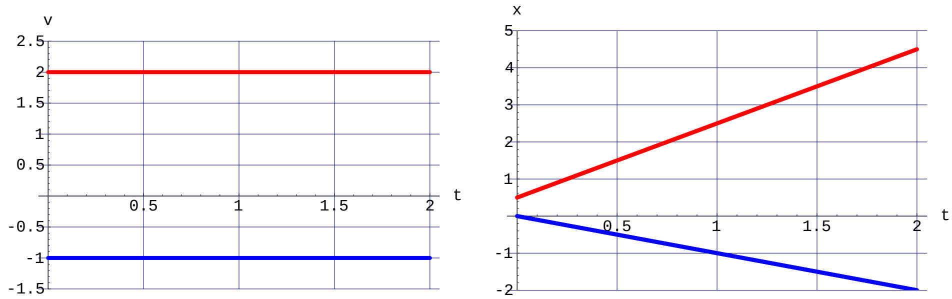


FIG. 5: Velocity (left) and position (right) plots for motion with constant velocity: Positive (red) or negative (blue). Note that area under the velocity line (positive or negative) corresponds to the change in position: E.g. (red) $2 \times 2 = 4.5 - 0.5$, or (blue) $2 \times (-1) = -2 - 0$.

Displacement

$$\Delta x = v\, \Delta t \tag{5}$$

Distance

$$D = |\Delta x| \tag{6}$$

Speed

$$s = D/\Delta t = |v| \tag{7}$$

Example (trap!):

$$S_1 = 2\,km/h;\,,\ S_2 = 4\,km/h\ .\ \ S_a - ?$$

$$D = 2AB\,,\ t_1 = AB/S_1\,,\ t_2 = AB/S_2$$

$$S_a = \frac{D}{t_1 + t_2} = \frac{2AB}{AB/S_1 + AB/S_2} = \frac{2S_1S_2}{S_1 + S_2} \neq 3\,km/h$$

B. $v \neq const$

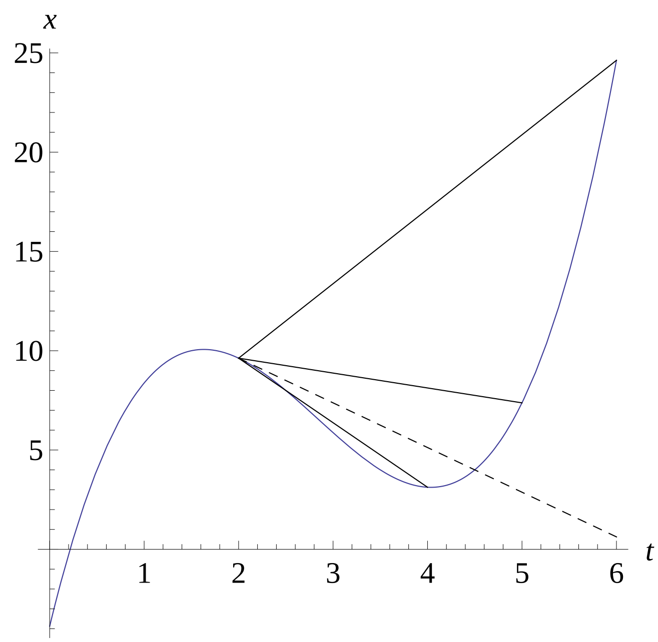


FIG. 6: A sample position vs. time plot (blue curve), and determination of the average velocities - slopes (positive or negative) of straight solid lines. Slope of dashed line (which is tangent to $x(t)$ curve) is the instantaneouus velocity at $t = 2$.

For explicit $x(t)$ plots for $v \neq const$ see fig. 8.

Average velocity:

$$v_{av} = \frac{\Delta x}{\Delta t} \tag{8}$$

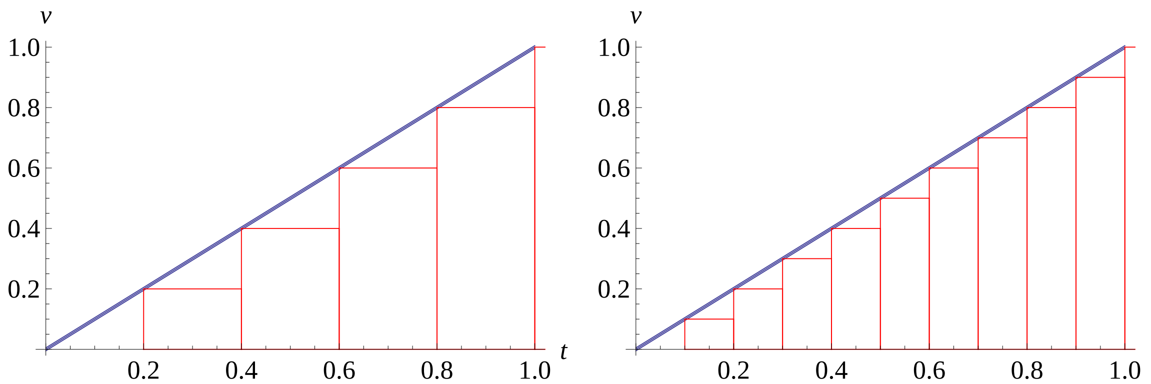


FIG. 7: Determination of displacement for a variable $v(t)$. During the ith small interval of duration Δt the velocity is replaced by a constant v_i shown by a horizontal red segment. Corresponding displacement is $\Delta x_i \approx v_i \cdot \Delta t$ (the red rectangular box). The total displacement $\Delta x = \sum \Delta x_i$ is then approximated by the area under the $v(t)$ curve. The error - the total area of the small triangles becomes small for small Δt (compare left and right figures) and vanishes in the strict limit $\Delta t \to 0$, when the sum becomes an integral.

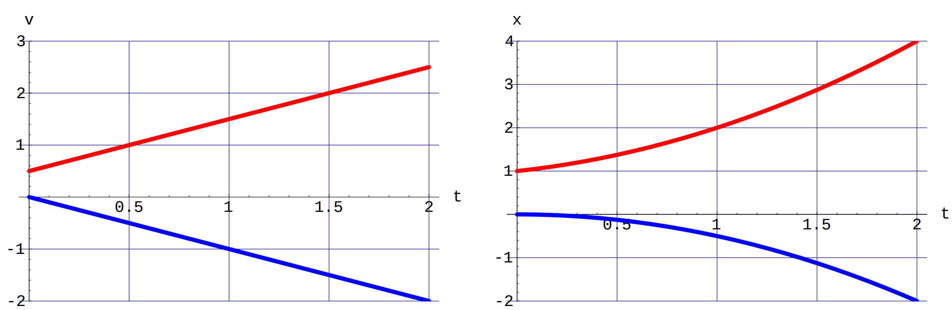


FIG. 8: Velocity (left) and position (right) plots for motion with constant acceleration: Positive (red) or negative (blue). Again, area under the velocity line (positive or negative) corresponds to the change in position. E.g. (red) $(2.5+0.5) \times 2/2 = 4-1$ or (blue) $(-2) \times 2/2 = -2-0$.

Average speed

$$s_{av} = \frac{D}{\Delta t} \geq |v_{av}| \tag{9}$$

Distance:

$$D \geq |\Delta x| \tag{10}$$

Instantaneous velocity

$$v = \lim_{\Delta t \to 0} v_{av} = \frac{dx}{dt} \tag{11}$$

Displacement Δx - area under the $v(t)$ curve or *Advanced*

$$\Delta x(t) = \int_{t_1}^{t} v\left(t'\right)\, dt'$$

Acceleration

$$a_{av} = \frac{\Delta v}{\Delta t} \tag{12}$$

$$a = \lim_{\Delta t \to 0} a_{av} = \frac{dv}{dt} = \frac{d^2 x}{dt^2} \tag{13}$$

C. $a = const$

Notations: Start from $t = 0$, thus $\Delta t = t$; $v(0) \equiv v_0$.

$$\Delta v = at \tag{14}$$

or

$$v = v_0 + at$$

Displacement - area of the trapezoid in fig. 8 (can be negative!):

$$\Delta x = \frac{v_0 + v}{2} t = v_0 t + at^2/2 \tag{15}$$

A useful alternative: use $t = (v - v_0)/a$:

$$\Delta x = \frac{v_0 + v}{2}\frac{v - v_0}{a} = \frac{v^2 - v_0^2}{2a} \tag{16}$$

(A more elegant derivation follows from conservation of energy,... later)

SUMMARY: if

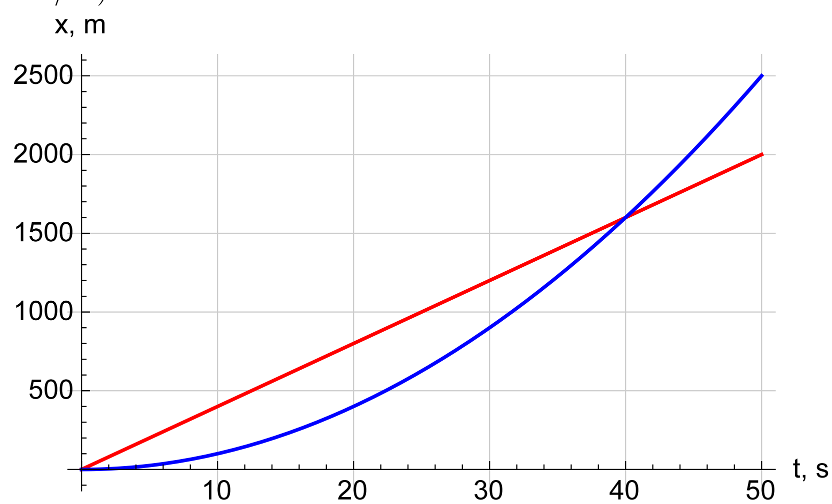
$$a = const \tag{17}$$

$$v = v_0 + at \tag{18}$$

$$x = x_0 + v_0 t + \frac{1}{2}at^2 \tag{19}$$

$$x - x_0 = \frac{v_0 + v}{2}t = \frac{v^2 - v_0^2}{2a} \tag{20}$$

Examples: meeting problems (car $V_C = 40\,m/s$, $a_C = 0$ and motorcycle $V_M = 0$, $a_M = 2\,m/s^2$)



$$X_C = V_C t\,,\ X_M = \frac{1}{2}a_M t^2$$

$$V_C t = \frac{1}{2}a_M t^2\,,\ t = 2V_C/a_M = 40\,s\,,\ X_{meet} = t \cdot V_C = 1600\,s = \frac{1}{2}a_M t^2$$

D. Free fall

(a)

Reminder: if

$$a = const \tag{21}$$

$$v = v_0 + at \tag{22}$$

$$x = x_0 + v_0 t + \frac{1}{2} a t^2 \tag{23}$$

$$x - x_0 = \frac{v_0 + v}{2} t = \frac{v^2 - v_0^2}{2a} \tag{24}$$

Free fall:

$$a \rightarrow -g \, , \; x \rightarrow y \, , \; x_0 \rightarrow y_0 \; (or, \, H)$$

$$a = -g = -9.8 \, m/s^2 \tag{25}$$

$$v = v_0 - gt \tag{26}$$

$$y = y_0 + v_0 t - \frac{1}{2} g t^2 \tag{27}$$

$$y - y_0 = \frac{v_0^2 - v^2}{2g} \tag{28}$$

Example: max height:

$$v = 0 \, , \; y_{\max} - y_0 = v_0^2/2g \tag{29}$$

Example: the Tower of Piza ($v_0 = 0 \, , \; y_0 = H \simeq 55 \, m$).

$$0 = H + 0t - gt^2/2 \, , \; t = \sqrt{\frac{2H}{g}}$$

$$0 - H = -v^2/2g \, , \; v = \sqrt{2gH}$$

Advanced. Exact evaluation of partial sums in a specific example.

Consider fig.7. Let $v = at$. Break the time t into N intervals with $\Delta t = t/N$ being the horizontal size of each rectangle. Then, number each rectangle by i, with $0 \leq i \leq N - 1$.

The vertical size of a rectangle is then $\frac{i}{N}at$, so that the total area covered by all rectangles is

$$\sum_{i=0}^{N-1} at\frac{i}{N}\Delta t = at\frac{\Delta t}{N}\sum_{i=0}^{N-1} i =$$

$$= at\frac{\Delta t}{N}\frac{N(N-1)}{2} = \frac{at^2}{2}\left(1 - \frac{1}{N}\right)$$

The $1/N$ term (area of the triangles in fig.7) indeed vanishes as $N \to \infty$.

IV. 2D MOTION

A. Introduction: Derivatives of a vector

$$\vec{r}(t) = (x(t)\,,\ y(t)) = x(t)\vec{i} + y(t)\vec{j} \tag{30}$$

$$\frac{d}{dt}\vec{r} = \left(\frac{dx}{dt}\,,\ \frac{dy}{dt}\right) = \frac{dx}{dt}\vec{i} + \frac{dy}{dt}\vec{j} \tag{31}$$

$$\frac{d^2}{dt^2}\vec{r} = \left(\frac{d^2x}{dt^2}\,,\ \frac{d^2y}{dt^2}\right) = \frac{d^2x}{dt^2}\vec{i} + \frac{d^2y}{dt^2}\vec{j} \tag{32}$$

B. General

Position:

$$\vec{r} = \vec{r}(t) \tag{33}$$

Average velocity:

$$\vec{v}_{av} = \frac{\Delta\vec{r}}{\Delta t} \tag{34}$$

(see Fig. 9).

Instantaneous velocity:

$$\vec{v} = \lim_{\Delta t\to 0} \vec{v}_{av} = \frac{d\vec{r}}{dt} \tag{35}$$

Average acceleration:

$$\vec{a}_{av} = \frac{\Delta\vec{v}}{\Delta t} \tag{36}$$

Instantaneous acceleration:

$$\vec{a} = \lim_{\Delta t \to 0} \vec{a}_{av} = \frac{d\vec{v}}{dt} = \frac{d^2\vec{r}}{dt^2} \tag{37}$$

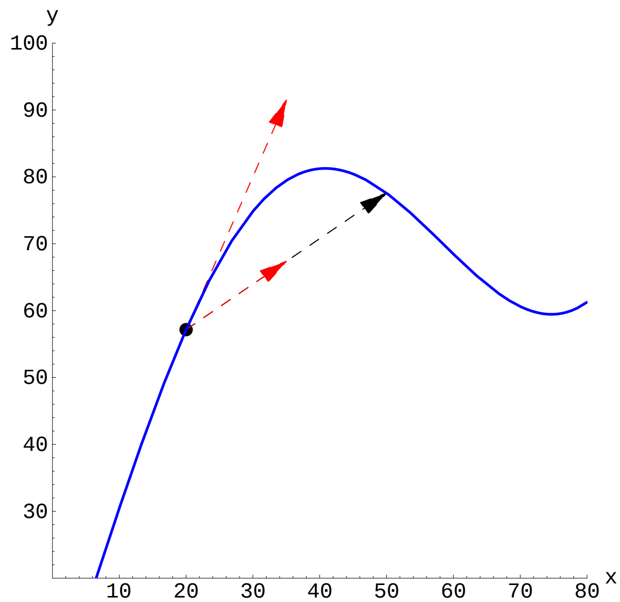


FIG. 9: Position of a particle $\vec{r}(t)$ (blue line), finite displacement $\Delta\vec{r}$ (black dashed line) and the average velocity $\vec{v} = \Delta\vec{r}/\Delta t$ (red dashed in the same direction). The the instantaneous velocity at a given point is tangent to the trajectory.

C. $\vec{a} = const$

$$\Delta\vec{v} = \vec{a} \cdot \Delta t$$

or with $t_0 = 0$

$$\vec{v} = \vec{v}_0 + \vec{a} \cdot t \tag{38}$$

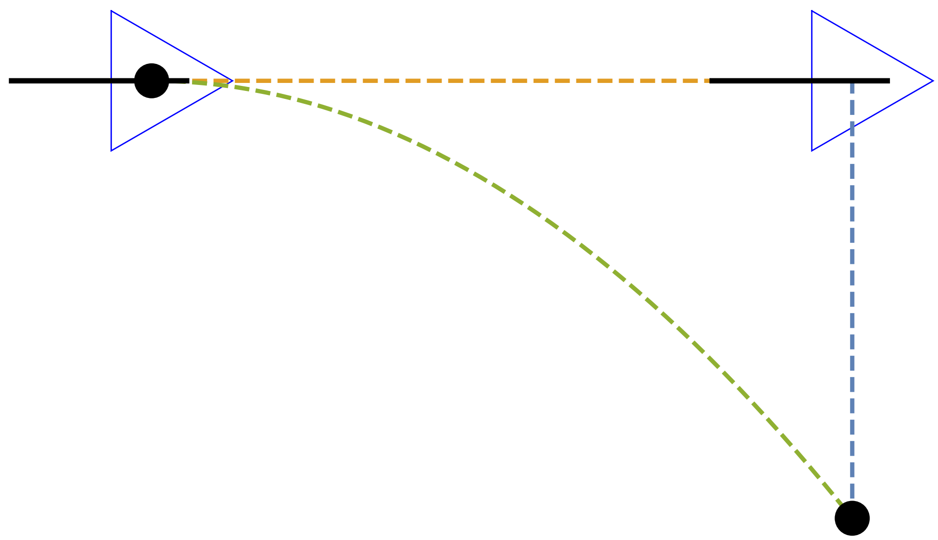
Displacement:

$$\vec{r} = \vec{r}_0 + \vec{v}_0 \cdot t + \frac{1}{2}\vec{a} \cdot t^2 \tag{39}$$

(The above can be proven either by integration or by writing eq. (38) in components and using known 1D results).

D. $\vec{a} = \vec{g}$ (projectile motion)

1. Introduction: Object from a plane



Given: H, V (horizontal). Find: L, v upon impact.

Distance. From y-direction

$$t = \sqrt{2H/g}$$

From x-direction

$$L = Vt = \dots$$

Speed upon impact. From y-direction

$$v_y^2 = v_{0y}^2 + 2gH$$

From x-direction

$$v_x = const = V$$

$$v^2 = v_x^2 + v_y^2 = v_x^2 + v_{0y}^2 + 2gH = v_0^2 + 2gH$$

Angle of impact with horizontal:

$$\tan\theta = v_y/v_x = -\sqrt{2gH}/V$$

General: Select x-axis horizontal, y-axis vertical.

$$a_x = 0\ , \quad a_y = -g \tag{40}$$

From eq. (38) written in components one has

$$v_x = v_{0,x} = const\ , \quad v_y = v_{0,y} - gt \tag{41}$$

with

$$v_{0,x} = v_0\cos\theta\ , \quad v_{0,y} = v_0\sin\theta \tag{42}$$

Displacement:

$$x = x_0 + v_{0,x}t \tag{43}$$

$$y = y_0 + v_{0,y}t - \frac{1}{2}gt^2 \tag{44}$$

Note:

$$y_{\max} - y_0 = \frac{v_{0,y}^2}{2g} \tag{45}$$

$$x_{\max} = \frac{v_{0,x}v_{0,y}}{g} \tag{46}$$

Range:

$$R = 2x_{\max} = \frac{v_0^2}{g}\sin(2\theta) \tag{47}$$

Note maximum for $\theta = \pi/4$.

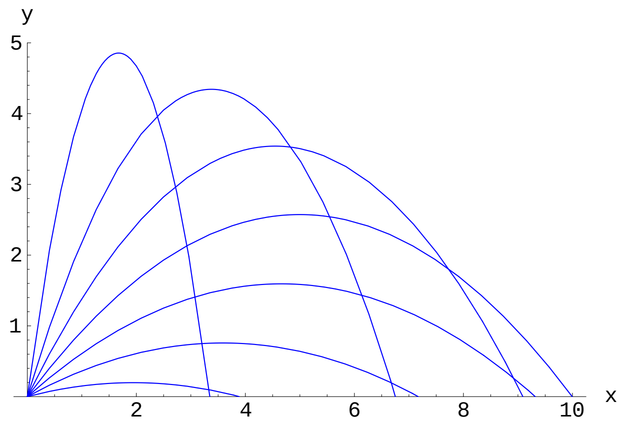


FIG. 10: Projectile motion for different values of the initial angle θ with a fixed value of initial speed v_0 (close to $10\,m/s$). Maximum range is achieved for $\theta = \pi/4$.

Trajectory: (use $x_0 = y_0 = 0$). Exclude time, e.g. $t = x/v_{0,x}$. Then

$$y = x\frac{v_{0,y}}{v_{0,x}} - \frac{1}{2}g\frac{x^2}{v_{0,x}^2} = x\tan\theta - \frac{g}{2v_0^2\cos^2\theta}x^2 \tag{48}$$

This is a parabola - see Fig. 10.

Problem. A coastguard cannon is placed on a cliff $y_0 = 60\,m$ above the sea level. A shell is fired at an angle $\theta = 30^o$ above horizontal with initial speed $v_0 = 80\,m/s$. Find the following:

1. the horizontal distance x from the cliff to the point where the projectile hits the water
2. the speed upon impact

Solution:

$$v_x = v_0\cos\theta\,,\;\; v_{0y} = v_0\sin\theta$$

Note $y = 0$ at the end. Find time from vertical motion only

$$0 = y_0 + v_{0y}t - \frac{1}{2}gt^2$$

(a quadratic equation - select a positive root). Then, the horizontal distance

$$x = v_x t$$

Speed upon impact (just as for the plane)

$$v = \sqrt{v_0^2 + 2gy_0} > v_0$$

Angle θ does not affect the final speed.

E. Uniform circular motion

1. *Preliminaries*

Consider motion sround a circle with a constant speed v. The velocity $\vec{v}$, however, changes directions so that there is acceleration.

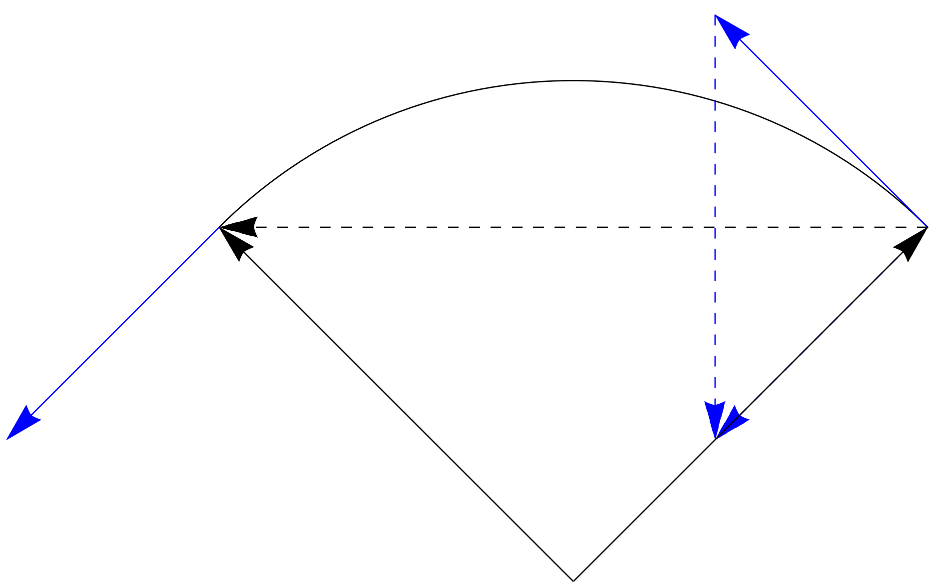
Period of revolution:

$$T = 2\pi r/v \tag{49}$$

with $1/T$ - "frequency of revolution". Angular velocity (in rad/s):

$$\omega = 2\pi/T = v/r \tag{50}$$

2. *Acceleration*



Note that $\vec{v}$ is *always* perpendicular to $\vec{r}$. Thus, from geometry vectors $\vec{v}\,(t+\Delta t)\,,\vec{v}(t)$ and $\Delta\vec{v}$ form a triangle which is similar to the one formed by $\vec{r}\,(t+\Delta t)\,,\vec{r}(t)$ and $\Delta\vec{r}$. Or,

$$\frac{|\Delta\vec{v}|}{v} = \frac{|\Delta\vec{r}|}{r}$$

$$a = \lim_{\Delta t\to 0}\frac{|\Delta\vec{v}|}{\Delta t} = \frac{v}{r}\lim_{\Delta t\to 0}\frac{|\Delta\vec{r}|}{\Delta t}$$

Or

$$a = v^2/r = \omega^2 r \tag{51}$$

3. *An alternative derivation*

We can use derivatives with the major relation

$$\frac{d}{dt}\sin(\omega t) = \omega\cos(\omega t)\ , \quad \frac{d}{dt}\cos(\omega t) = -\omega\sin(\omega t)\ , \tag{52}$$

One has

$$\vec{r}(t) = (x,\ y) = (r\cos\omega t,\ r\sin\omega t)$$
$$\vec{v}(t) = \frac{d\vec{r}}{dt} = (-r\omega\sin\omega t,\ r\omega\cos\omega t)$$

$$\vec{a} = \frac{d\vec{v}}{dt} = \left(-r\omega^2\cos\omega t,\ -r\omega^2\sin\omega t\right) = -\omega^2\vec{r} \tag{53}$$

which gives not only magnitude but also the direction of acceleration opposite to $\vec{r}$, i.e. towards the center.

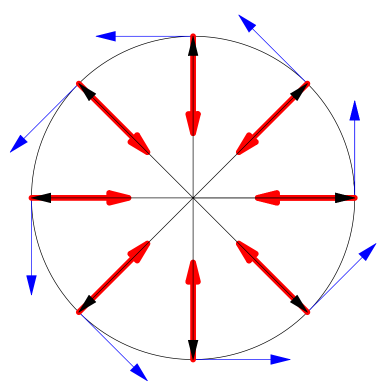


FIG. 11: Position (black), velocities (blue) and acceleration (red) vectors for a uniform circular motion in counter-clockwise direction.

F. Advanced: Classical (Galileo's) Relativity

New reference frame:

$$\vec{R} = \vec{R}_0 + \vec{V} t \, , \; \vec{V} = const \tag{54}$$

New coordinates, etc.:

$$\vec{r}' = \vec{r} - \vec{R} = \vec{r} - \vec{V} t - \vec{R}_0 \tag{55}$$

$$\vec{v}' = \vec{v} - \vec{V} \tag{56}$$

Example: Projectile with $v_x \neq 0$ and new ref. frame with $\vec{V} = (v_x \, , \, 0)$.

V. NEWTON'S LAWS

A. Force

1. Units

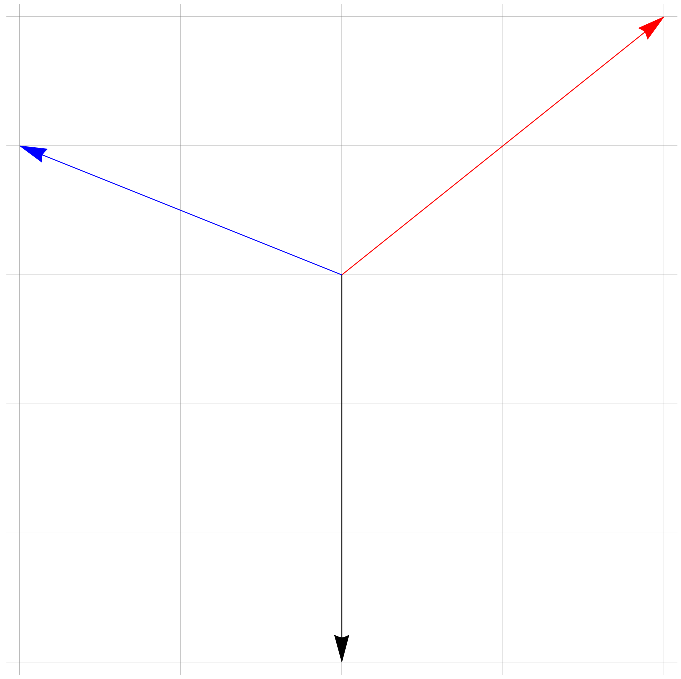
"Newton of force"

$$N = kg\frac{m}{s^2} \tag{57}$$

2. Vector nature

From experiment action of two independent forces $\vec{F}_1$ and $\vec{F}_2$ is equivalent to action of the resultant

$$\vec{F} = \vec{F}_1 + \vec{F}_2 \tag{58}$$



$$\vec{F}_1 + \vec{F}_2 + \vec{F}_3 = 0$$

$$\vec{F}_3 = -\left(\vec{F}_1 + \vec{F}_2\right)$$

B. The Laws

1. If $\vec{F} = 0$ (no net force) then $\vec{v} = const$

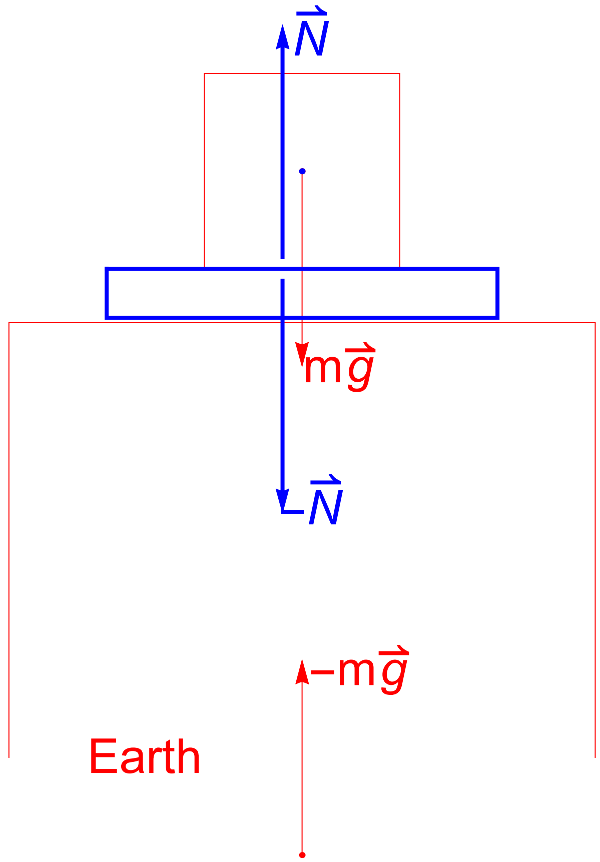
2.

$$\vec{F} = m\vec{a} \tag{59}$$

3.

$$\vec{F}_{12} = -\vec{F}_{21} \tag{60}$$

Notes: the 1st Law is *not* a trivial consequence of the 2nd one for $\vec{F} = 0$, but rather it identifies inertial reference frames where a free body moves with a constant velocity. In the 3rd Law forces $\vec{F}_{12}$ and $\vec{F}_{21}$ are applied to *different* bodies; both forces, however, are of the same physical nature (e.g. gravitational attraction).



Examples. Force from equiliibrium or from acceleration (without specifying the nature of force).

- resultant of two forces (in class)
- A force F acts on a particle with mass m which accelerates from rest over the distance of Δx during time t. Find F. *Solution.* From $\Delta x = \frac{1}{2}at^2$ find $a = 2\Delta x/t^2$; then, $F = ma$.
- A car with mass M increases speed from v_1 to v_2 over distance Δx. Find the force. *Solution.* From $\Delta x = \frac{1}{2}(v_1 + v_2)\, t$ find t. Then, $a = (v_2 - v_1)/t$ and $F = Ma$. (Alternatively, can use $\Delta x = (v_2^2 - v_1^2)/(2a)$ to find a

directly.)

1. Gravitational force

$$\vec{F}_g = m\vec{g} \tag{61}$$

Weight - the force an object excerts on a support. Will conside with $m\vec{g}$ for a stationary object. (Note: the textbook uses the latter as a definiton with weight always equal $m\vec{g}$; other books will use the former definiton of "weight", or "apparent weight" as force on a support. That can be different from $m\vec{g}$ in case of acceleration, with the extreme example of "weightlessness", as will be discussed in class).

Example: Gravitational plus another force.

A rocket of mass m starts from rest and accelerates up due to a force F from the engine. At the altitude H the engine shuts off. Describe the full motion.

With engine:

$$a_1 = (F - mg)/m$$

Speed at H from

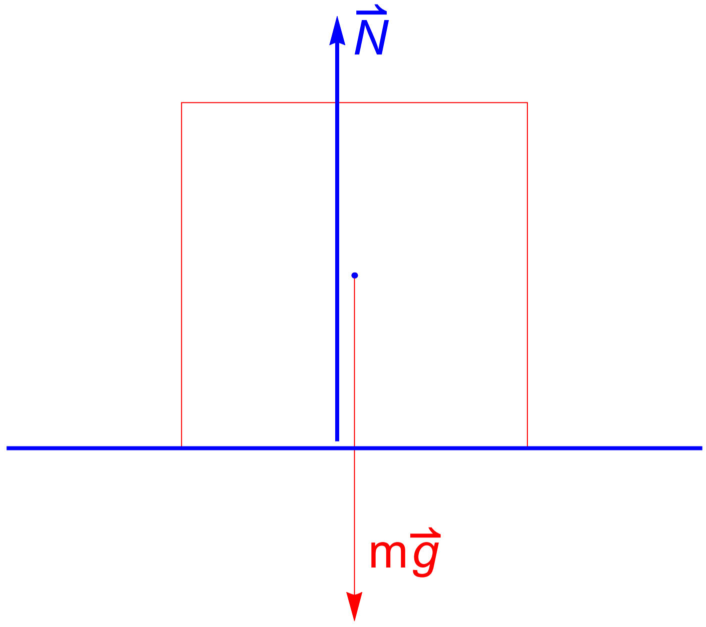
$$H = (v^2 - 0)/(2a_1)$$

Without engine

$$a_2 = -g$$

and extra h from $h = v^2/(2g)$

2. FBD, normal force



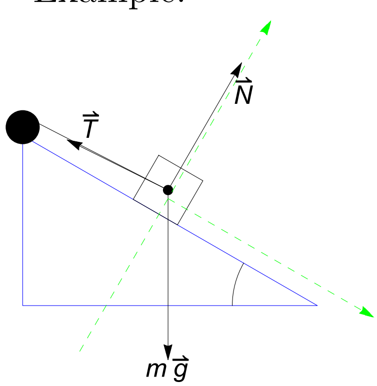
C. Statics

In equilibrium

$$\vec{F} \equiv \sum_i \vec{F}_i = 0 \tag{62}$$

(all forces are applied to the same body!).

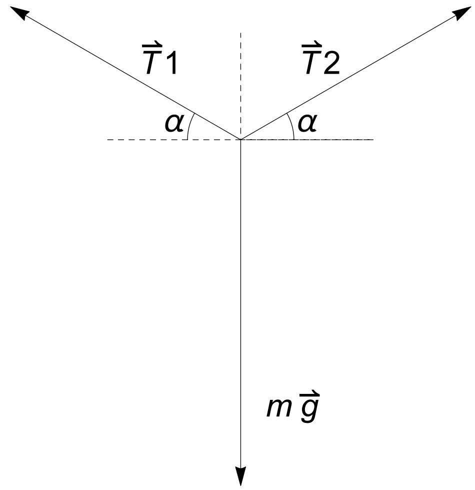
Example:



$$\vec{T} + \vec{N} + m\vec{g} = 0$$

$$x) : \quad -T + mg\sin\theta = 0\,, \; T = mg\sin\theta$$

$$y): \quad N - mg\cos\theta = 0$$



Example: From symmetry

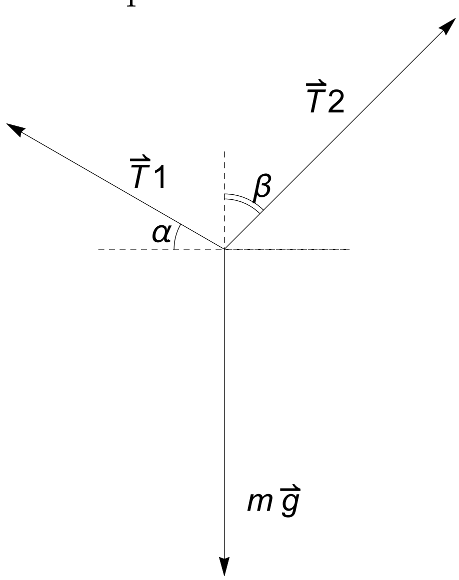
$$|\vec{T}_1| = |\vec{T}_2| \equiv T$$

x - automatic; from y:

$$T\sin\alpha + T\sin\alpha - mg = 0\,,\ T = mg/(2\sin\alpha)$$

Note the limits!

Example:



$$\vec{T}_1 + \vec{T}_2 + m\vec{g} == 0$$

$$-T_1\cos\alpha + T_2\sin\beta = 0$$

$$T_1 \sin\alpha + T_2 \cos\beta - mg = 0$$

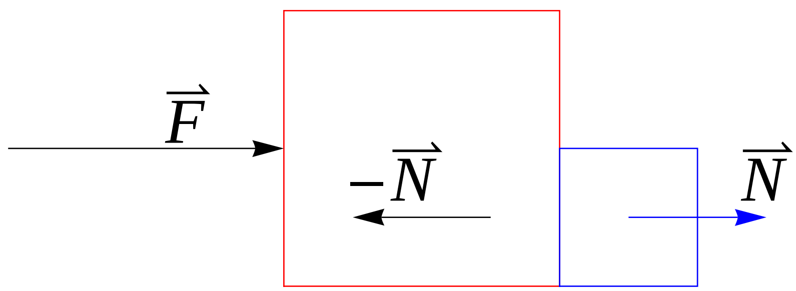
Then use $T_2 = T_1 \cos\alpha / \sin\beta$...

D. Dynamics: Examples

The following examples will be discusses in class:

- finding force between two accelerating blocks
- apparent weight in an elevator
- Atwood machine
- block on inclined plane (no friction)

Two blocks: a horizontal force $\vec{F}$ is applied to a block with mass M (red) which in turn pushes a block with mass m (blue); ignore friction. Find $\vec{N}$ and $\vec{a}$. (note that the 3rd Law is already used in the diagram).



Solution (quick): first treat the 2 blocks as a single solid body. From 2nd Law for this "body"

$$\vec{a} = \vec{F}/(M+m)$$

From 2nd Law for mass m:

$$\vec{N} = m\vec{a}$$

Solution (academic): write 2nd Law for each of the 2 block separately. For M

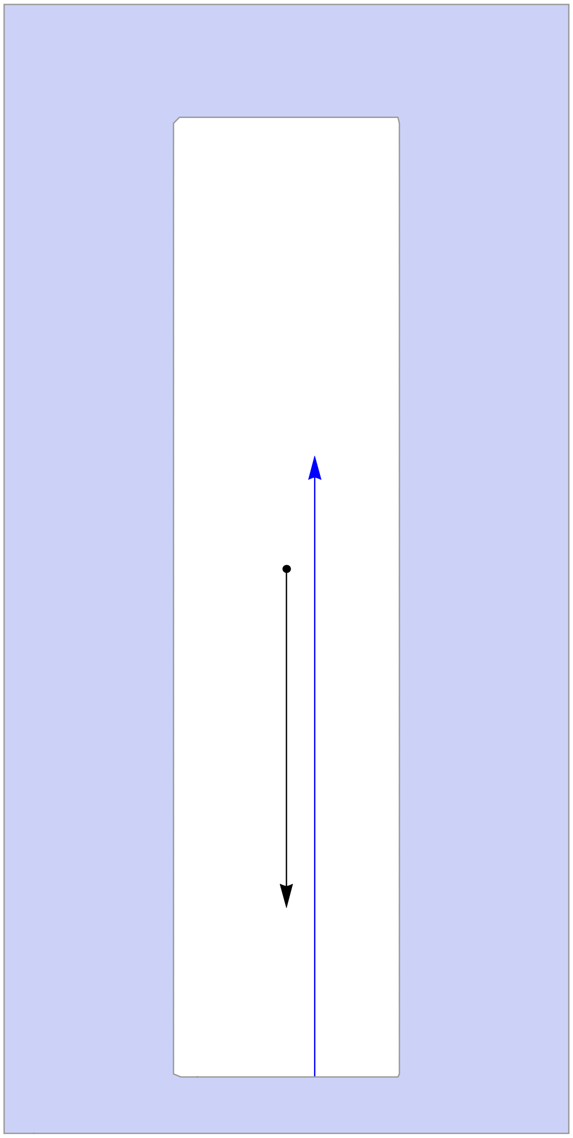
$$\vec{F} + \left(-\vec{N}\right) = M\vec{a}$$

For m, as above

$$\vec{N} = m\vec{a}$$

which will lead to the same result (add the 2 equations together to find $\vec{a}$).

Weight in an elevator: Find the "apparent weight" of a person of mass m if the elevator accelerates up with acceleration $\vec{a}$.



Solution. Forces on the person: $m\vec{g}$ (down, black, applied to center-of-mass) and $\vec{N}$ (reaction of the floor, blue, applied to feet) - up. [A force $-\vec{N}$ -not shown- acts on the floor and determines "apparent weight"]. The 2nd Law for the person

$$m\vec{g} + \vec{N} = m\vec{a}$$

From here

$$\vec{N} = m\left(\vec{a} - \vec{g}\right)$$

or in projection on vertical axis

$$N = m(g + a)$$

Note: $N > mg$ for $\vec{a}$ up and $N < mg$ for $\vec{a}$ down; $N = 0$ for $\vec{a} = \vec{g}$.

Atwood machine:

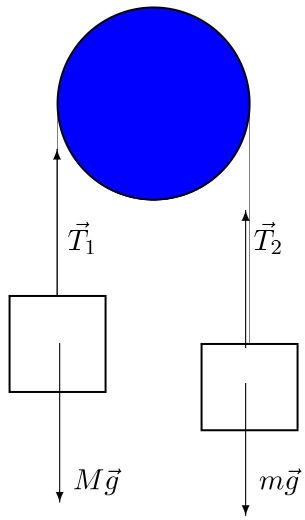


FIG. 12: Atwood machine. Mass M (left) is almost balanced by a slightly smaller mass m. Pulley has negligible rotational inertia, so that both strings have the same tension $\left|\vec{T}_1\right| = \left|\vec{T}_2\right| = T$

Let T_1 and T_2 be tensions in left string (connected to larger mass M) and in the right string, respectively.

- 2nd Law(s) for each body; for M axis down, for m - up.
- constrains: a-same, tensions - same

From 2nd Law(s):

$$Mg - T_1 = Ma\,,\ T_2 - mg = ma$$

Add together to get (with constrains)

$$(M - m)g = (M + m)a$$

which gives

$$a = g\frac{M-m}{M+m}$$

What if need tension?

$$T_1 = Mg - ma \; < \; Mg \; , \;\; T_2 = mg + ma \; > \; mg$$

(should be the same).

Consider the last example - Fig. 13. First let us solve it in several major steps and then try to come up with general suggestions.

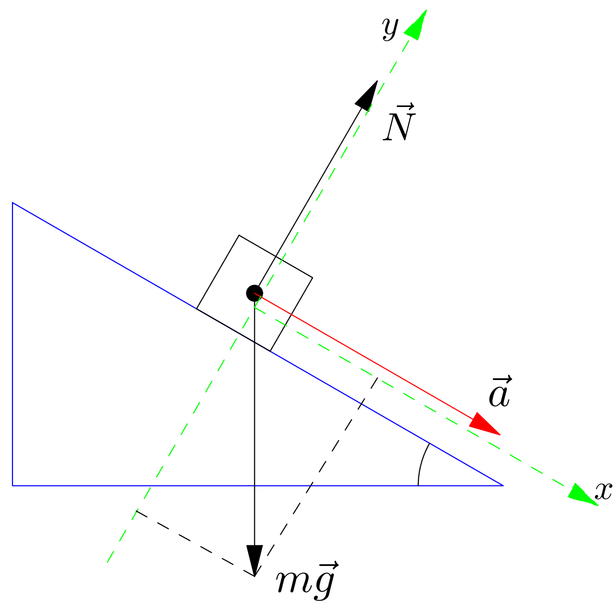


FIG. 13: A block on a frictionless inclined plane which makes an angle θ with horizontal.

- identify forces, $m\vec{g}$ and $\vec{N}$ in our case and the assumed acceleration $\vec{a}$ (magnitude still to be found).

- write the 2'd Law (vector form!)

$$\vec{N} + m\vec{g} = m\vec{a}$$

- select a "clever" system of coordinates x, y.

- write down projections of the vector equation on the x, y axes, respectively:

$$x: \quad mg \sin\theta = ma$$

$$y: \quad N - mg \cos\theta = 0$$

  the x-equation will give acceleration $a = g \sin\theta$ (which is already the solution); the y equation determines N.

- before plugging in numbers, a good idea is to check the limits. Indeed, for $\theta = 0$ (horizontal plane) $a = 0$ no acceleration, while for $\theta = \pi/2$ one has $a = g$, as should be for a free fall.

A few practical remarks to succeed in such problems.

- The original diagram should be BIG and clear. If so, you will use it as a FBD, otherwise you will have to re-draw it separately with an extra possibility of mistake.

- In the picture be realistic when dealing with "magic" angles of 30, 45, 60 and 90 degrees. Otherwise, a clear picture is more important than a true-to-life angle.

- Vectors of forces should be more distinct than anything else in the picture; *do not* draw arrows for projection of forces - they can be confused with real forces if there are many of them.

- the force of gravity in the picture should be immediately identified as $m\vec{g}$ (using an extra tautological definiton, such as $\vec{F}_g = m\vec{g}$ adds an equation and confuses the picture).

- If only one body is of interest, do not draw any forces which act on other bodies (in our case that would be, e.g. a force $-\vec{N}$ which acts on the inclined plane).

- select axes only *after* the diagram is completed and the 2'd Law is written in vector form. As a rule, in dynamic problems one axes is selected in the direction of acceleration (if this direction can be guessed).

VI. NEWTON'S LAWS: APPLICATIONS TO FRICTION AND TO CIRCULAR MOTION

A. Force of friction

Force of friction on a moving body:

$$f = \mu N \tag{63}$$

Direction - against velocity; μ (or μ_k) - kinetic friction coefficient.

Static friction:

$$f_s \leq \mu_s N \tag{64}$$

with μ_s - static friction coefficient.

1. Example: block on inclined plane

(see next page)

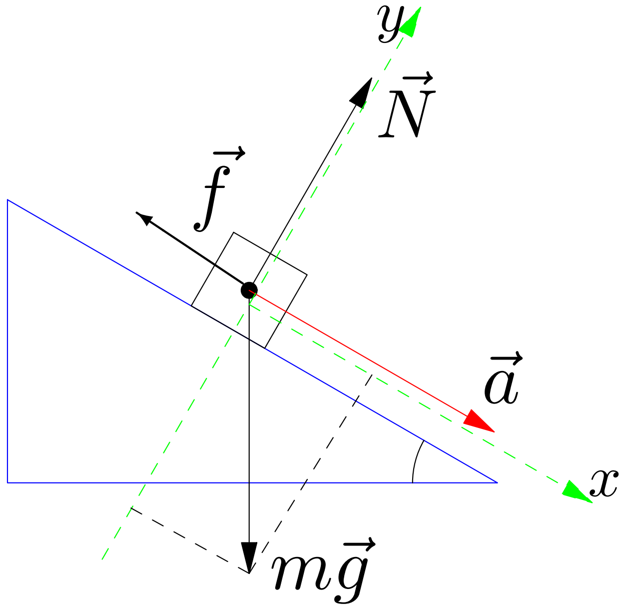


FIG. 14: A block sliding down an inclined plane with friction. The force of friction $\vec{f}$ is opposite to the direction of motion and equals $\mu_k N$.

- identify forces, $\vec{f}$, $m\vec{g}$ and $\vec{N}$ and the acceleration $\vec{a}$ (magnitude still to be found).

- write the 2'd Law (vector form!)

$$\vec{f} + \vec{N} + m\vec{g} = m\vec{a}$$

- select a "clever" system of coordinates x, y.

- write down projections of the vector equation on the x, y axes, respectively:

$$x: \quad -f + mg\sin\theta = ma$$

$$y: \quad N - mg\cos\theta = 0$$

- relate friction to normal force:

$$f = \mu_k N$$

the above 3 equations can be solved (in class). Answer:

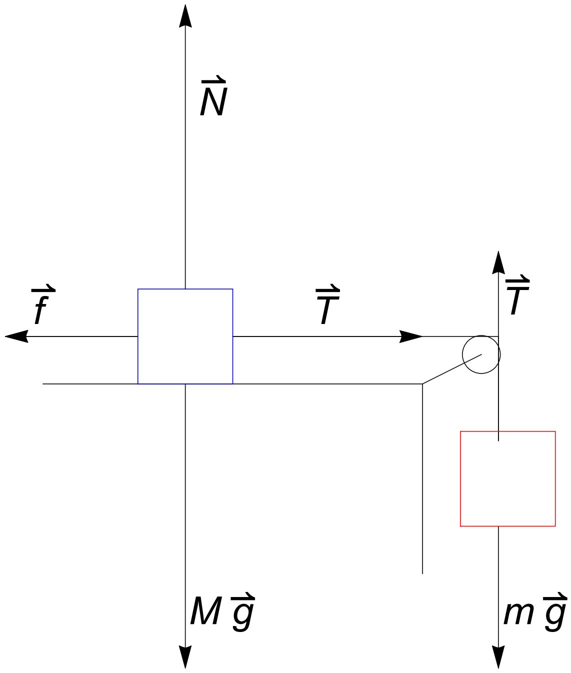
$$a = g\,(\sin\theta - \mu_k \cos\theta)$$

- Note: the body will keep moving only for

$$\mu_k < \tan\theta$$

The body will *start* moving only for

$$\mu_s < \tan\theta$$



First, try without friction, i.e. $f = 0$ and do not need $\vec{N}$ and $\vec{M}g$.

- for body M on the table

$$T = Ma$$

- for hanging body m:

$$mg - T = ma$$

Thus, adding the two equations together (to get rid of T):

$$mg = Ma + ma\,,\ or\ a = g\frac{m}{m+M}$$

With frition:

- for body M on the table

$$T - f = Ma\ (horizontal)$$

$$N - Mg = 0\ (vertical)$$

- for hanging body m (same as before):

$$mg - T = ma$$

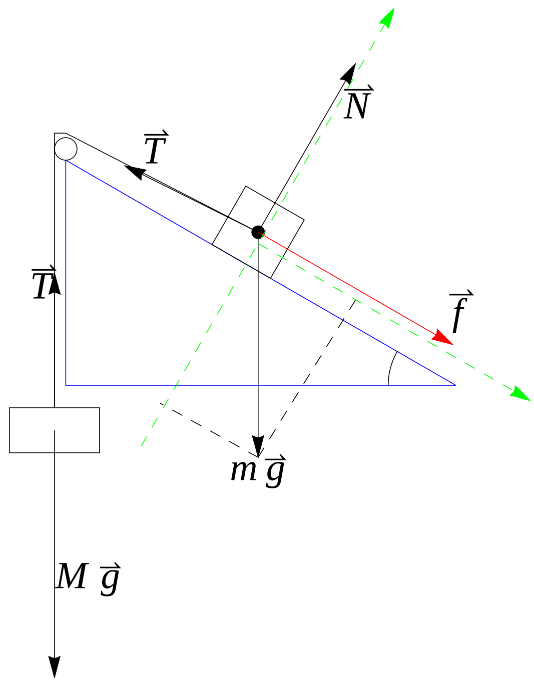
-

$$f = \mu N = \mu Mg\ \ (new)$$

Thus,

$$mg - f = (M+m)a \ , \quad a = \frac{mg - \mu M g}{M+m} = g\frac{m - \mu M}{M+m} \ > 0$$

(if $\mu M > m$ mootion is impossiblle - friction too strong).



Assume large enough M to move left. Hanging mass (axis down):

$$Mg - T = Ma \tag{65}$$

Mass on incline (x-axis uphill):

$$T - mg\sin\theta - f = ma \tag{66}$$

From above

$$g(M - m\sin\theta) - f = (M+m)a \tag{67}$$

How to find friction? - from N:

$$N - mg\cos\theta = 0 \tag{68}$$

$$N = mg\cos\theta\,,\;\; f = \mu N \tag{69}$$

$$g(M - m\sin\theta - \mu m\cos\theta) = (M+m)a \tag{70}$$

$$a = \ldots \tag{71}$$

B. Fast way to solve quasi-one-dimensional problems

Only external forces or their projections along the motion. No tension, no normal force. Friction

$$f = \mu mg\ (horizontal)\ , \quad f = \mu mg \cos\theta\ (inclined)$$

General

$$\sum F_{ext} = M_{tot} a$$

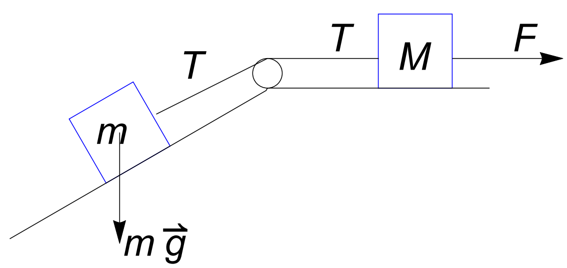
Example: Atwood machine

$$Mg - mg = (m + M)a\,,\ a = \ldots$$

if need T

$$T - mg = ma\,,\ T = \ldots$$

Example: external force (no friction)

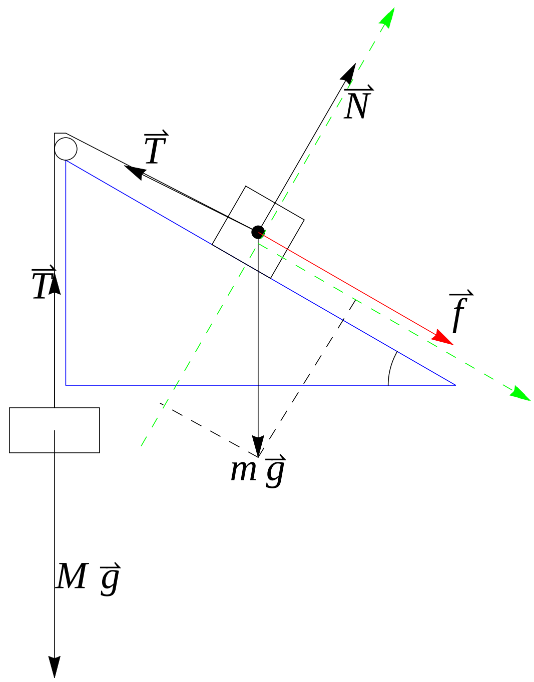


$$F - mg\sin\theta = (M + m)a\,,\ a = \ldots$$

Tension (if need) from

$$F - T = Ma\,,\ T = F - Ma$$

Example: 2 blocks with friction, large M

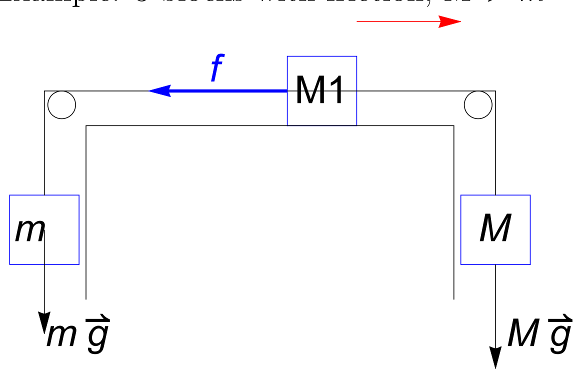


select positive direction uphill

$$Mg - \mu mg\cos\theta - mg\sin\theta = (M+m)a\,,\; a = \ldots$$

$$Mg - T = Ma\,,\; T = M(g-a)$$

Example: 3 blocks with friction, $M > m$



$$Mg - mg - \mu M_1 g = (m + M_1 + M)a\,,\; a = \ldots$$

Tension T_1 between M_1 and M from

$$Mg - T_1 = Ma$$

Tension T between M_1 and m from

$$T - mg = ma$$

C. Centripetal force

Newton's laws are applicable to *any* motion, centripetal including. Thus, for centripetal force of any physical origin

$$F_c = ma_c \equiv m\frac{v^2}{r} = m\omega^2 r \tag{72}$$

Centripetal force is always directed towards center, perpendicular to the velocity - see Fig. 15. Examples: tension of a string, gravitational force, friction.

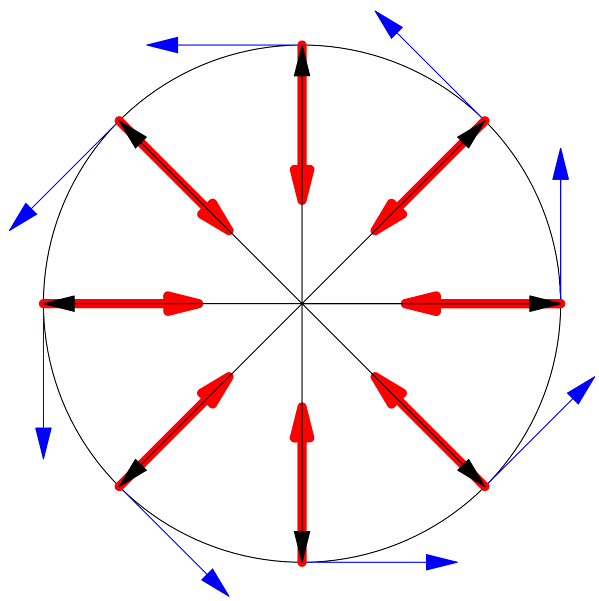


FIG. 15: Position (black), velocities (blue) and centripetal force (red) vectors for a uniform circular motion in counter-clockwise direction. The direction of force coincides with centripetal acceleration (towards the center). The value of centripetal force at each point is determined by the vector sum of actual physical forces, e.g. normal force and gravity in case of a Ferris Wheel, or tension plus gravity in case of a conic pendulum.

Simple example: A particlle with mass around a circle with a diameter $d = 2\,m$. It takes $t = 3\,s$ to complete one revolution. Find the magnitude of the centripetal force F.
Solution: Since $F = ma_c$, need acceleration. Use

$$a_c = v^2/r \; with \; v = \pi d/t \; and \; a_c = 2\pi^2\frac{d}{t^2}$$

Equivalently

$$a_c = \omega^2 \frac{d}{2} \; with \; \omega = 2\pi/t$$

Thus,

$$F = 2\pi^2 \frac{md}{t^2} = \ldots$$

1. Conic pendulum

See Fig. 16, forces will be labeled in class.

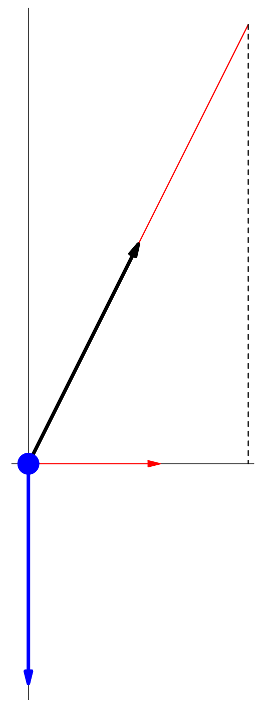


FIG. 16: A conic pendulum rotating around a vertical axes (dashed line). Two forces gravity (blue) and tension (black) create a centripetal acceleration (red) directed towards the center of rotation. The angle with vertical is θ, length of the s tring is L and radius of revolution is $r = L \sin\theta$.

One has the 2nd Law:

$$\vec{T} + m\vec{g} = m\vec{a}_c$$

In projections:

$$x : \quad T \sin\theta = ma_c = m\omega^2 r$$

$$y : \quad T \cos\theta - mg = 0$$

Thus,

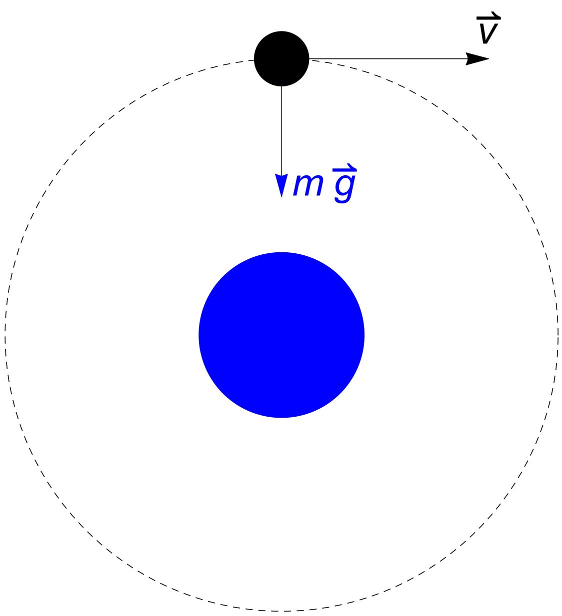
$$\omega = \sqrt{\frac{g}{L\cos\theta}} \simeq \sqrt{\frac{g}{L}}$$

(the approximation is valid for $\theta \ll 1$). The period of revolution

$$\frac{2\pi}{\omega} \simeq 2\pi\sqrt{\frac{L}{g}}$$

Note that mass and angle (if small) do not matter.

2. Satellite



$$\vec{F}_g = m\vec{a}_c$$

$$F_g = mg\,,\ a_c = v^2/r$$

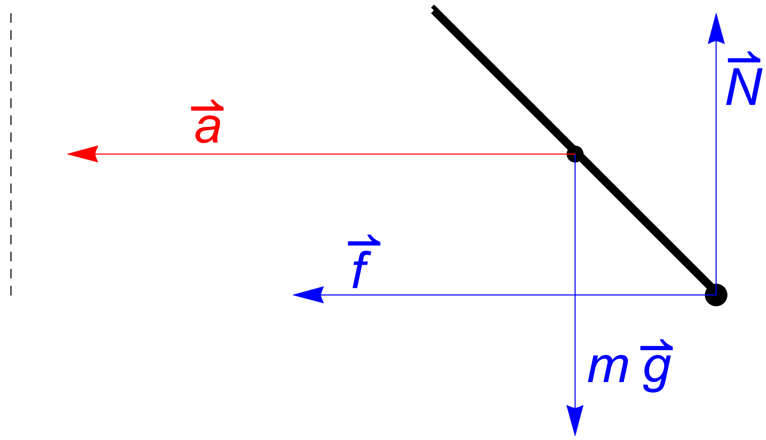
Thus,

$$g = \frac{v^2}{r}$$

$$v = \sqrt{gr}$$

about $8\,km/s$ for Earth. Period of revolution

$$2\pi\sqrt{r/g}$$



$$\vec{N} + \vec{f} + m\vec{g} = m\vec{a}$$

x-axis - towards the center (dashed):

$$f = ma = mv^2/R$$

y-up:

$$N - mg = 0$$

and

$$f \leq \mu_s N = \mu_s mg$$

Thus,

$$v^2/R \leq \mu_s g$$

D. (Advanced) "Forces of inertia"

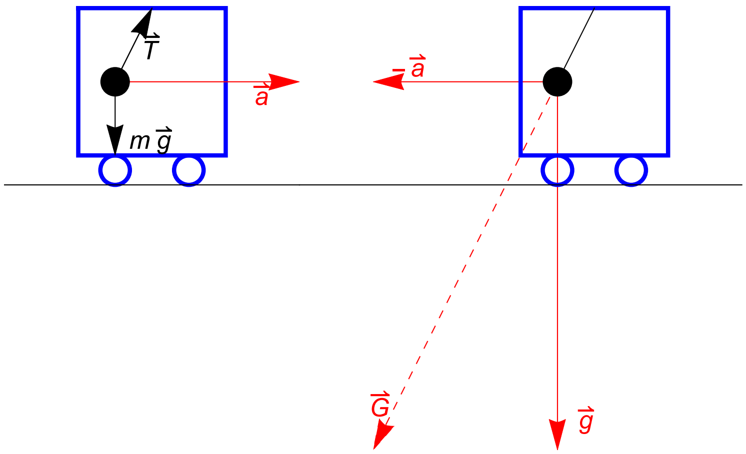
$$\vec{F} = m\vec{a}$$

Define

$$\vec{F}_i = -m\vec{a}$$

$$\vec{F} + \vec{F}_i = 0$$

"statics".



$\vec{T}$
$\vec{a}$
$m\vec{g}$
$-\vec{a}$
$\vec{G}$
$\vec{g}$

VII. WORK

A. Units

Joule (J):

$$1\,J = 1\,N\cdot m = kg\frac{m^2}{s^2} \tag{73}$$

B. Definitions

Constant force:

$$W = \vec{F}\cdot\Delta\vec{r} \equiv F_x\Delta x + F_y\Delta y \tag{74}$$

Example: work by force of gravity

$$F_x = 0\ ,\ \ F_y = -mg$$

$$\boxed{W_g = -mg\Delta y} \tag{75}$$

(and Δx absolutely does not matter!)

Example:

$$\vec{F} = \vec{i} + 2\vec{j}\ ,\ \vec{r}_1 = 3\vec{i} + 4\vec{j}\ ,\ \vec{r}_2 = 6\vec{i} - 4\vec{j}$$

$$\Delta\vec{r} = 3\vec{i} - 8\vec{j}\ ,\ W = 3\cdot 1 + (-8)\cdot 2 = -13$$

Variable force:

Let us break the path from $\vec{r}_1$ to $\vec{r}_2$ in small segments $\Delta\vec{r}_i$ with $\vec{F}_i \simeq const$. Then

$$W_i \simeq \vec{F}_i\cdot\Delta\vec{r}_i$$

and total work

$$W = \sum_i W_i \;\to\; \int_{\vec{r}_1}^{\vec{r}_2} \vec{F} \cdot d\vec{r} \tag{76}$$

Note: forces which are perpendicular to displacement do not work, e.g. the centripetal force, normal force.

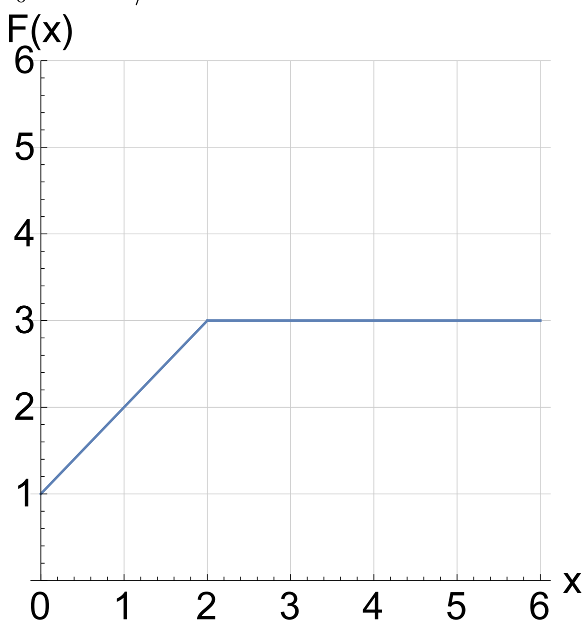
C. 1D motion and examples

For motion in x-direction only

$$W = \int_{x_1}^{x_2} F_x \, dx \tag{77}$$

If F_x is given by a graph, work is the area under the curve (can be negative!).

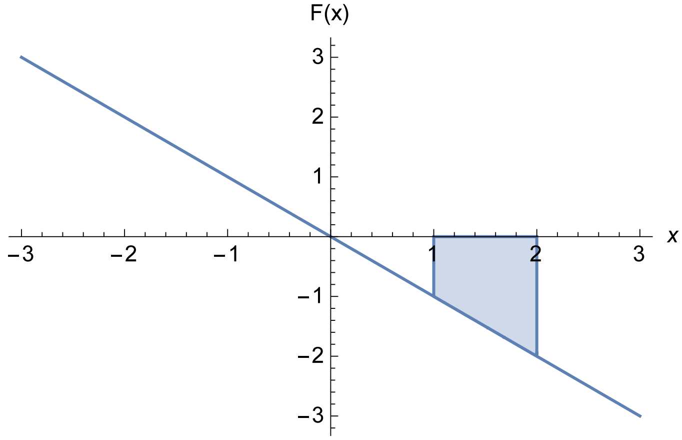
Example: find work and (from KET) the final speed of an $m = 2\,kg$ particle with $v_0 = 10\,m/s$



$$W = 16\,J\,,\; W = \frac{1}{2}mv^2 - \frac{1}{2}mv_0^2$$

$$v^2 = v_0^2 + 2W/m = \ldots$$

Spring:



$$F = -kx \tag{78}$$

k - "spring constant" (also known as "Hook's law"). Work done by the spring

$$W_{sp} = \frac{1}{2}(F_2 + F_1)(x_2 - x_1) = -\frac{1}{2}kx_2^2 + \frac{1}{2}kx_1^2 \tag{79}$$

(If the spring is stretched from rest, the term with x_1^2 will be absent and the work done by the spring will be negative for any $x_2 \neq 0$.)

VIII. KINETIC ENERGY

A. Definition and units

$$K = \frac{1}{2}mv^2 \tag{80}$$

or if many particles, the sum of individual energies.

Units: J (same as work).

B. Relation to work

1. *Constant force*

$$W = F_x\Delta x + F_y\Delta y = m\left[a_x\Delta x + a_y\Delta y\right]$$

According to kinematics

$$a_x \Delta x = \frac{v_x^2 - v_{0x}^2}{2} \ , \ \ a_y \Delta y = \frac{v_y^2 - v_{0y}^2}{2}$$

Thus,

$$W = \Delta K \tag{81}$$

which is the "work-energy" theorem.

Examples: gravity and friction (in class).

The "policeman problem". Given: L, μ, find v_o

Friction: $f = \mu m g$.

$$W = -fL = -\mu mgL \ < 0 \ (!) \tag{82}$$

$$\Delta K = 0 - \frac{1}{2} m v_0^2 \tag{83}$$

$$-m v_0^2 / 2 = -\mu m g L \tag{84}$$

$$v = \sqrt{2 \mu g L} \tag{85}$$

2. Variable force

Again, break the displacement path into a large number N of small segments Δr_i . For each

$$W_i = \Delta K_i \equiv K_i - K_{i-1}$$

Thus,

$$W = \sum_i^N W_i = K_N - K_0 \equiv \Delta K$$

Example: spring.

$$\frac{1}{2} m \left(v^2 - v_0^2\right) = -\frac{k}{2} \left(x^2 - x_0^2\right)$$

C. Power

$$P = \frac{dW}{dt} = \vec{F} \cdot \vec{v}$$

Units: *Watts*. $1W = 1\,J/s$

Re-derivation of work-energy theorem:

$$\frac{d}{dt}K = \frac{d}{dt}\frac{m}{2}v^2 = \frac{m}{2}\frac{d}{dt}\left(\vec{v} \cdot \vec{v}\right) = m\frac{d\vec{v}}{dt} \cdot \vec{v} = \vec{F} \cdot \vec{v} = \frac{dW}{dt}$$

Example: An $M = 500\,kg$ horse is running up an $\alpha = 30^o$ slope with $v = 4\,m/s$. Find P:

$$P = mg\sin\alpha \cdot v \simeq 500 \cdot 9.8\frac{1}{2}4 \approx \ldots$$

Example. For a fast bike the air resistance is $\sim v^2$. How much more power is needed to double v? Ans.: 8 times more.

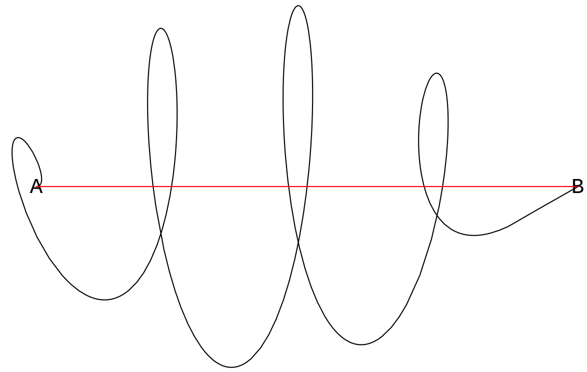


FIG. 17: Generally, the work of a force between points A and B depends on the actual path. However, for some "magic" (conservative) forces the work is path-independent. For such forces one can introduce potential energy U and determine work along *any* path as $W = U_A - U_B = -\Delta U$.

Dr. Vitaly A. Shneidman, Phys 111, 7th Lecture

IX. POTENTIAL ENERGY

A. Some remarkable forces with path-independent work

See Fig. 17 and caption.

Examples:

- Constant force

$$W = \sum \vec{F} \cdot \Delta \vec{r}_i = \vec{F} \cdot \sum \Delta \vec{r}_i = \vec{F} \cdot \Delta r = \vec{F} \cdot \vec{r}_B - \vec{F} \cdot \vec{r}_A \tag{86}$$

with potential energy

$$U(\vec{r}) = -\vec{F} \cdot \vec{r} \tag{87}$$

Example force of gravity with $F_x = 0\,, F_y = -mg$ and

$$\boxed{U_g = mgh} \tag{88}$$

- Elastic (spring) force

$$W \simeq -\sum k \frac{x_i + x_{i+1}}{2}\left(x_{i+1} - x_i\right) = -\frac{k}{2}\sum x_{i+1}^2 - x_i^2 = -\frac{k}{2}\left(x_f^2 - x_i^2\right) \tag{89}$$

with potential energy

$$\boxed{U_{sp}(x) = \frac{1}{2}kx^2} \tag{90}$$

- Other: full force of gravity $F = -mg$ with r-dependent g, and any force which does not depend on velocity but depends only on distance from a center.
- Non-conservative: kinetic friction (depends on velocity, since poits against $\vec{v}$)

B. Relation to force

$$U(x) = -\int F(x)\,dx \tag{91}$$

$$F = -dU/dx \tag{92}$$

X. CONSERVATION OF ENERGY

Start with

$$\Delta K = W$$

If only conservative forces

$$W = -\Delta U \tag{93}$$

thus

$$\boxed{K + U = const \equiv E} \tag{94}$$

Examples:

- Maximum hight.

$$E = mgh + \frac{1}{2}mv^2$$

$$0 + \frac{1}{2}mv_0^2 = mgh_{\max} + 0$$

$$h_{\max} = \frac{v_0^2}{2g}$$

- Galileo's tower. Find speed upon impact

$$mgh + 0 = 0 + \frac{1}{2}mv^2$$

$$v^2 = 2gh$$

- Coastguard cannon (from recitation on projectiles). Fing speed upon impact

$$mgH + \frac{1}{2}mv_0^2 = 0 + \frac{1}{2}mv^2$$

$$v^2 = v_0^2 + 2gH$$

(note that the angle does not matter!).

- A spring with given k, m is stretched by X meters and released. Find $v_{\max}$.

$$E = \frac{1}{2}kx^2 + \frac{1}{2}mv^2$$

$$\frac{1}{2}kX^2 + 0 = 0 + \frac{1}{2}mv_{\max}^2$$

$$v_{\max} = X\sqrt{k/m}$$

A. Conservative plus non-conservative forces

For conservative forces introduce potential energy U, and then define the total *mechanical* energy

$$E_{mech} = K + U$$

One has

$$W = -\Delta U + W_{non-cons} \tag{95}$$

Then, from work-energy theorem

$$\boxed{\Delta E_{mech} = \Delta (K + U) = W_{non-cons}} \tag{96}$$

Examples:

- Galileo's Tower revisited (with friction/air resistance). Given: $H = 55\,m$, $m = 1\,kg$ and $W_f = -100\,J$ is lost to friction (i.e. the mechanical energy is lost, the thermal energy of the mass m and the air is increased by $+100\,J$). Find the speed upon impact.

$$\left(0 + \frac{1}{2}mv^2\right) - (mgH + 0) = W_f < 0$$

$$\frac{1}{2}mv^2 = mgh - |W_f|\,,\ v = \ldots$$

- Sliding crate (from recitation)) revisited. Given L, m, θ, μ find the speed v at the bottom.

$$h \equiv L\sin\theta$$

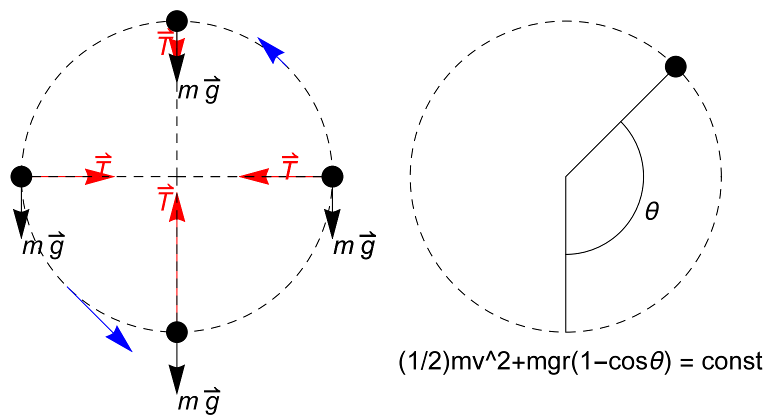
$$f = \mu mg\cos\theta$$

$$W_f = -fL$$

$$(0 + \frac{1}{2}mv^2) - (mgh + 0) = W_f$$

$$v^2 = 2gh - \frac{2}{m}|W_f|$$

Example: Problem which is too hard without energy



pendulum or one way rock-on-a-string?. Left: forces. Right: energy solution.

θ - angle with vertical. Then,

$$U_g = mgh = mgr\,(1 - \cos\theta)\ ,\ U_g^{\rm max} = 2mgr$$

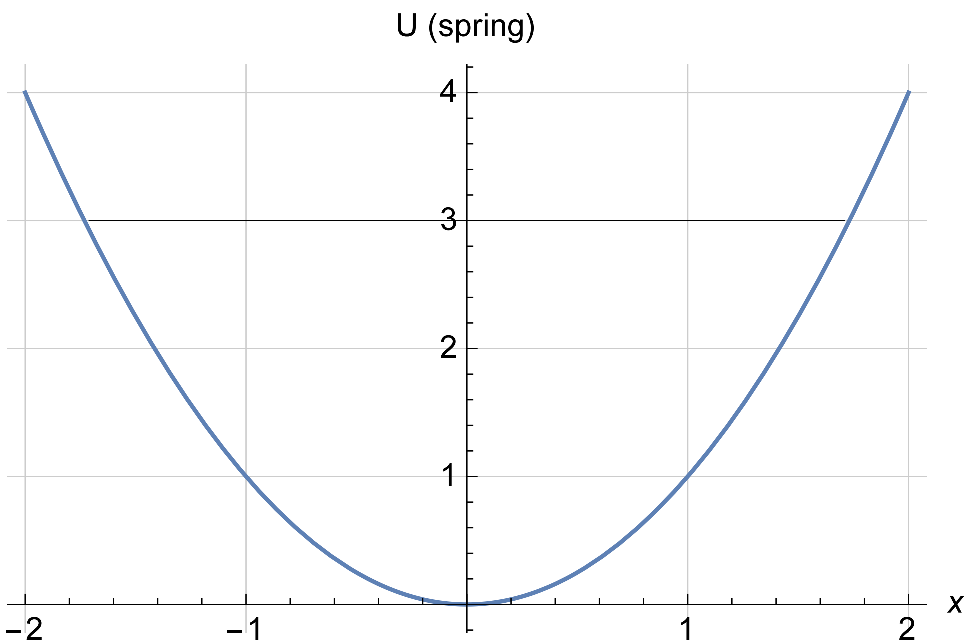
$$E = U_g + \frac{1}{2}mv^2\ ,\ E > 2mgr - rotation\ ,\ 0 < E < 2mgr - oscillations$$

What if need T? Lowest point

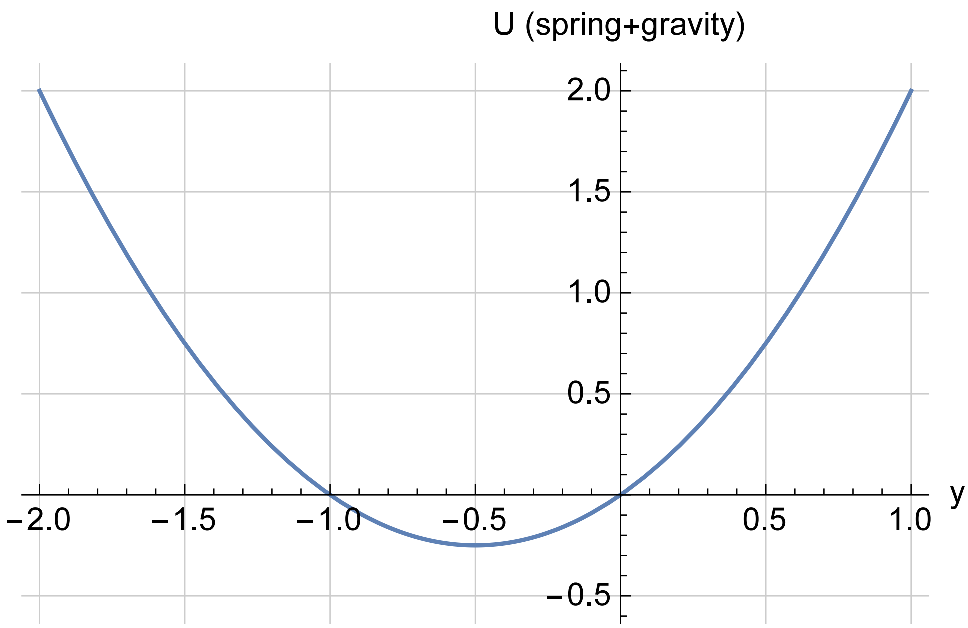
$$U_g = 0\ ,\ E = \frac{1}{2}mv^2$$

$$T - mg = mv^2/r\ =\ 2E/r$$

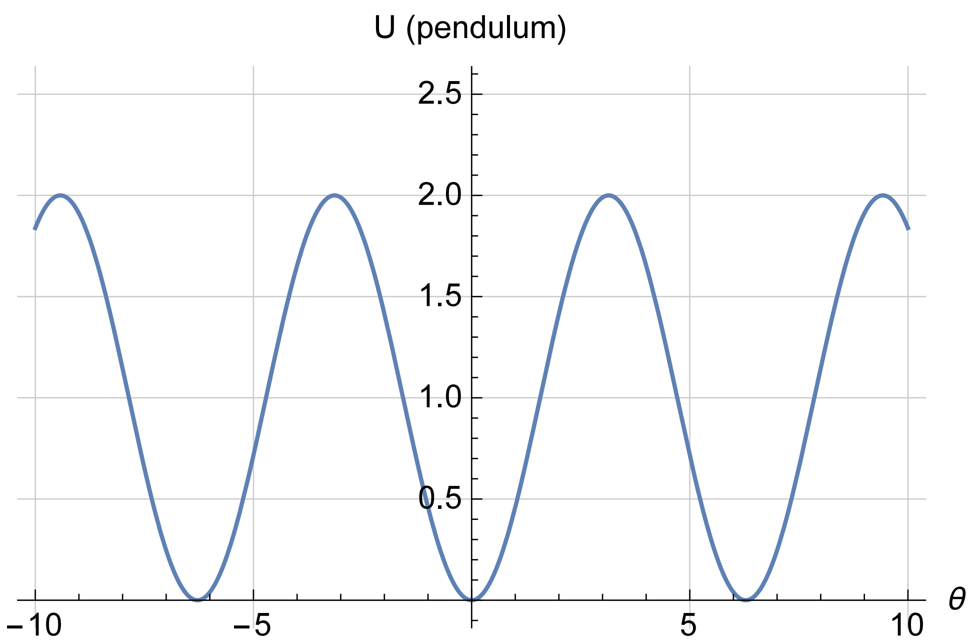
B. Advanced: Typical potential energy curves



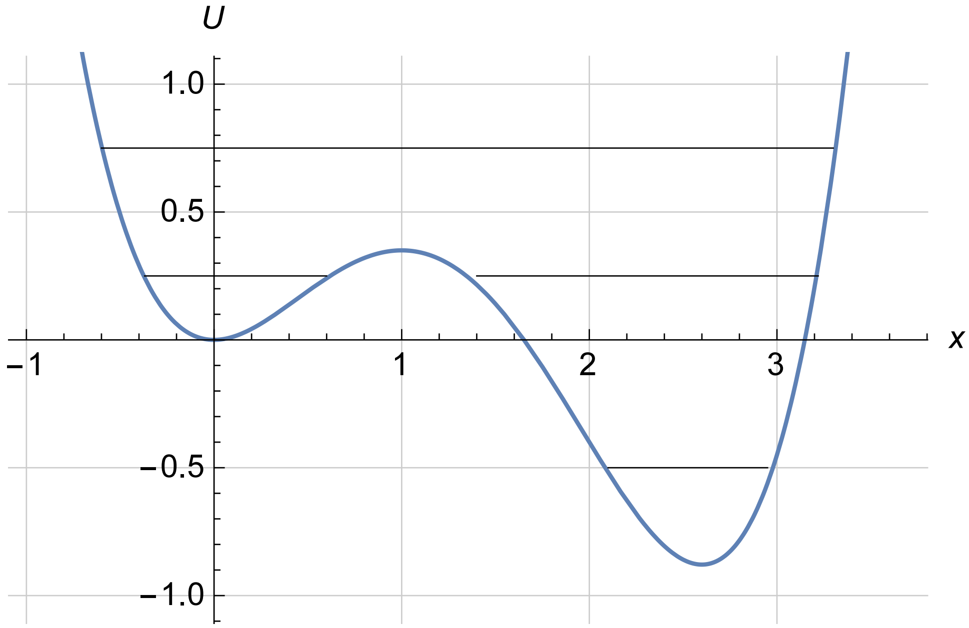
$$U = \frac{1}{2}kx^2 \tag{97}$$



$$U = \frac{1}{2}ky^2 + mgy \tag{98}$$

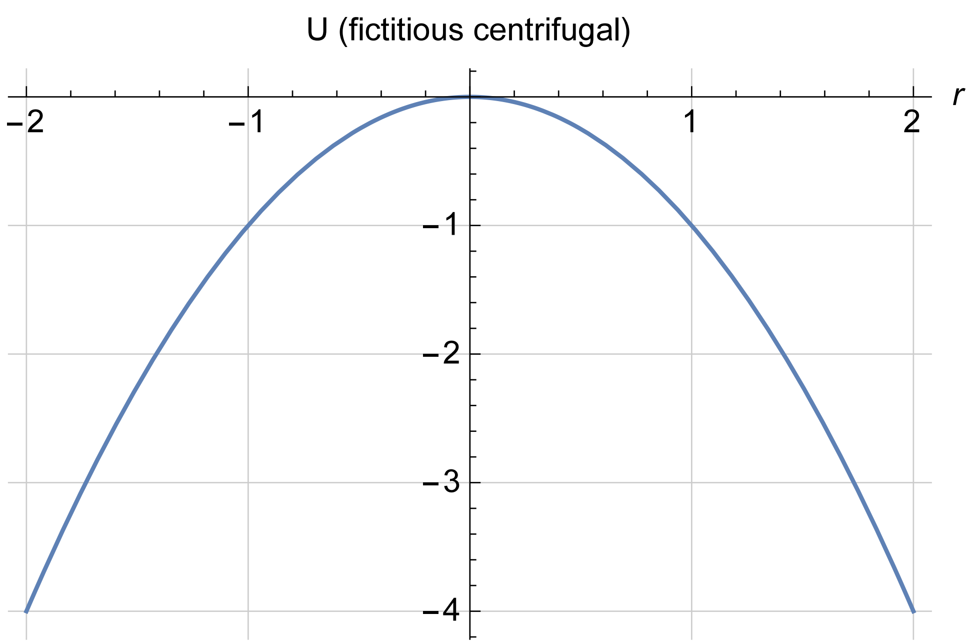


$$U = mgL(1 - \cos\theta) \tag{99}$$

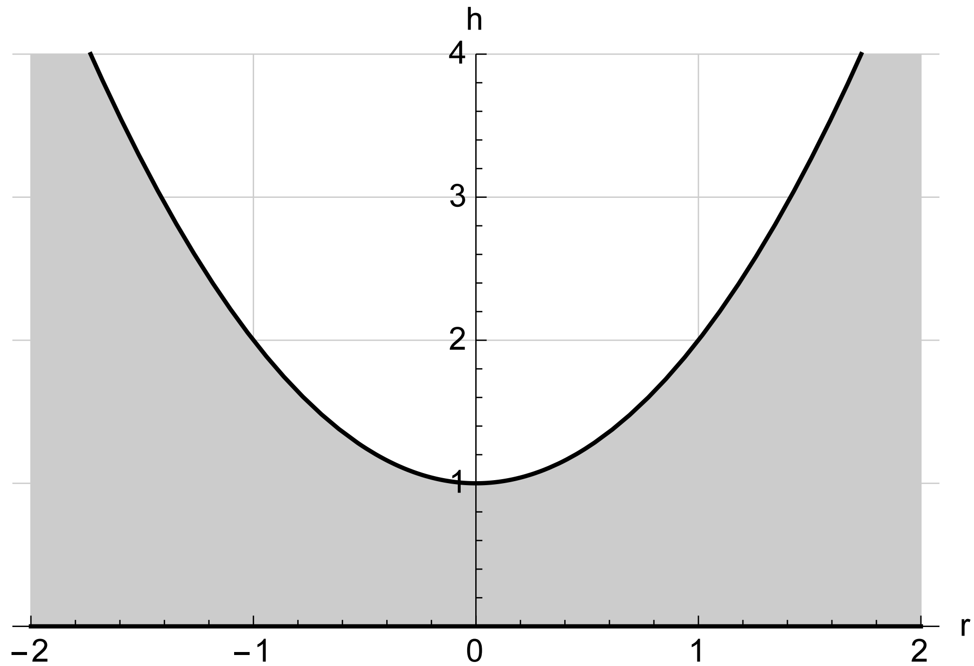


$$K = E - U \geq 0\,,\ v = \pm\sqrt{2K/m} \tag{100}$$

C. Advanced: Fictitious "centrifugal energy"



$$U = -\frac{1}{2}m\omega^2 r^2 \tag{101}$$



$$U = U_g + U_{cent} = mgh - \frac{1}{2}m\omega^2 r^2 = const \tag{102}$$

$$h(r) = h(0) + \frac{\omega^2}{g} r^2 \tag{103}$$

D. Advanced: Mathematical meaning of energy conservation

Start from 2nd Law with $F = F(x)$ (not v or t !)

$$m\ddot{x} = F(x) \mid \times \dot{x} \tag{104}$$

$$\dot{x}\ddot{x} = \frac{d}{dt}(\dot{x})^2/2$$
$$\dot{x}F(x) = \frac{d}{dt}\int F(x)\,dx = -\frac{d}{dt}U(x)$$

Thus

$$\frac{d}{dt}(K + U) = 0 \tag{105}$$

$$K + U = const = E \tag{106}$$

E. Examples:

1. Suppose, the crate from the previous recitation problem is let got at the top of the ramp and slides back to the floor. Do the following:

1. find the work done by the force of friction

$$W_f = -\mu mg \cos\theta\, L$$

2. find the change in potential ennrgy due to gravity (watch for the sign!)

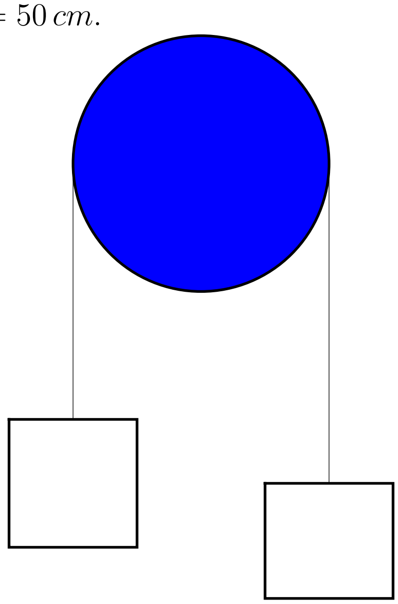
$$\Delta U = mg\,0 - mgh$$

3. find the speed at the bottom of ramp

$$(mv^2/2 - 0) + \Delta U = W_f$$

$$v = \ldots$$

2. In the Atwood machine the heavier body on left has mass $M = 1.001\,kg$, while the lighter body on right has mass $m = 1\,kg$. The system is initially at rest, there is no friction and the mass of the pulley can be ignored. Find the speed after the larger mass lowers by $h = 50\,cm$.



$$\Delta K = Mv^2/2 + mv^2/2 - 0$$

$$\Delta U = -Mgh + mgh$$

$$\Delta K + \Delta U = 0\,,\ v = \ldots$$

3. A skier slides down from a hill which is $H = 30\,m$ high and then, without losing speed, up a hill which is $h = 10\,m$ high. What is his final speed? (a) Ignore friction. (b) Assume a small average friction force of $1\,N$ and the combined length of the slopes $L = 200\,m$. The mass of the skier is $m = 80\,kg$.

$$\Delta K = mv^2/2\,,\ \Delta U = mgh - mgH$$

$$\Delta K + \Delta U = W_F = -fL\,,\ v = \ldots$$

4. A spring gun is loaded with a $m = 20\,g$ ball and installed vertically, with the zero level $y = 0$ corresponding to uncompressed spring. When loading the gun, the spring is compressed down by $10\,cm$. Find the maximum height reached by the ball after the gun is fired. The spring constant is $k = 100\,N/m$.

$$U_{sg} = ky^2/2 + mgy\,,\ y_0 = -10\,cm$$

$$U_{final} = mgh$$

$$\Delta K = 0$$

thus

$$\Delta U = 0$$

$$h = \ldots$$

XI. MOMENTUM

A. Definition

One particle:

$$\boxed{\vec{p} = m\vec{v}} \tag{107}$$

System:

$$\vec{\mathcal{P}} = \sum_i \vec{p}_i = \sum_i m_i \vec{v}_i \tag{108}$$

Units: $kg \cdot m/s$ (no special name).

B. 2nd Law in terms of momentum

Single particle:

$$m\vec{a} = m\frac{d\vec{v}}{dt} = \frac{d(m\vec{v})}{dt}$$

$$\frac{d\vec{p}}{dt} = \vec{F} \tag{109}$$

$$\Delta\vec{p} = \int_{t_1}^{t_2} dt\, \vec{F}(t) = \vec{F}_{av}\, \Delta t \tag{110}$$

$\int_{t_1}^{t_2} dt\, \vec{F}(t)$ - impulse. "It takes an impulse to change momentum".

Example. A steel ball with mass $m = 100\, g$ falls down on a horizontal plate with $v = 3\, m/s$ and rebounds with $V = 2\, m/s$ up. (a) find the impulse from the plate; (b) estimate F_{av} if the collision time is $1\, ms$.

Select the positive direction up. Then, $p_1 = -3 \times 0.1 = -.3\, kg\, m/s$ and after the collision

$p_2 = mV = 0.2\,kg\,m/s$. (a) Impulse $p_2 - p_1 = 0.5\,kg \cdot m/s$. (note: $|p_1|$ and $|p_2|$ add up!). (b) $F_{av} = 0.5/0.001 = 500\,N$ (large!).

System of two particles with NO external forces:

$$d\vec{p}_1/dt = \vec{F}_{12}$$

$$d\vec{p}_2/dt = \vec{F}_{21}$$

$$\frac{d\vec{\mathcal{P}}}{dt} = \vec{F}_{12} + \vec{F}_{21} = 0 \tag{111}$$

i.e.

$$\boxed{\vec{\mathcal{P}} = const} \tag{112}$$

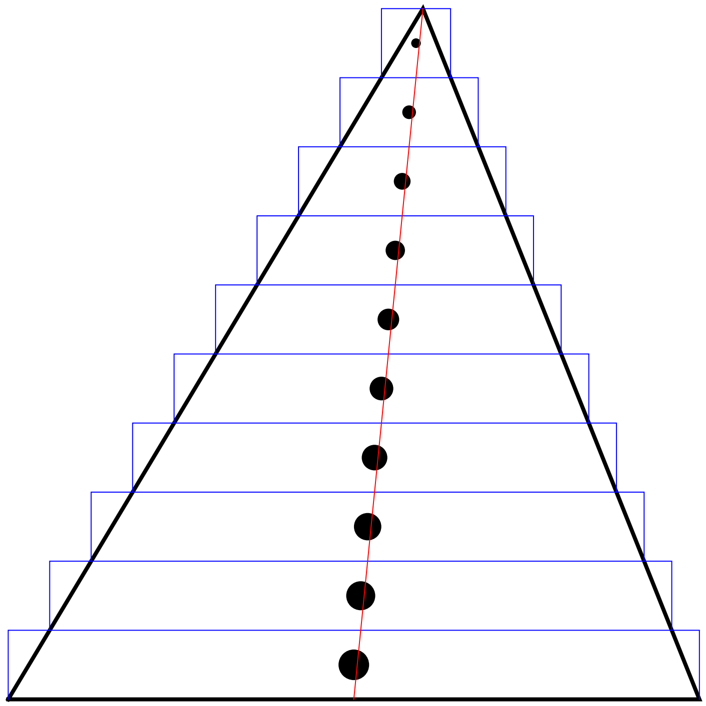
and the same for any number of particles.

XII. CENTER OF MASS (CM)

A. Definition

$$\vec{\mathcal{R}} = \frac{1}{M}\sum_i m_i \vec{r}_i \ , \quad M = \sum_i m_i \tag{113}$$

Example: CM of a triangle.

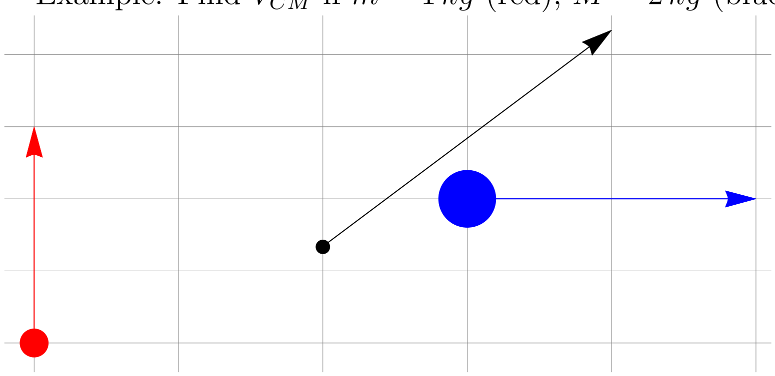


B. Relation to total momentum

$$\vec{V}_{cm} = \frac{d}{dt}\vec{\mathcal{R}} = \frac{1}{M}\sum_i m_i \frac{d}{dt}\vec{r}_i = \frac{1}{M}\vec{\mathcal{P}}$$

$$\boxed{\vec{\mathcal{P}} = M\vec{V}_{CM}} \tag{114}$$

Example: Find $\vec{V}_{CM}$ if $m = 1\,kg$ (red), $M = 2\,kg$ (blue), $\vec{v}_1 = (0,3)\,m/s$, $\vec{v}_2 = (2,0)\,m/s$



$$\vec{P} = (4,3)\,kg\,m/s\ , \quad \vec{V}_{CM} = (4/3,1)\,m/s$$

with

$$V_{CM} = 5/3\, m/s$$

C. 2nd Law for CM

Consider two particles:

$$d\vec{p}_1/dt = \vec{F}_{12} + \vec{F}_{1,ext}$$

$$d\vec{p}_2/dt = \vec{F}_{21} + \vec{F}_{2,ext}$$

$$\frac{d}{dt}\vec{\mathcal{P}} = \vec{F}_{1,ext} + \vec{F}_{2,ext} \equiv \vec{F}_{ext}$$

but

$$\frac{d}{dt}\vec{\mathcal{P}} = M\frac{d}{dt}\vec{V}_{cm} = M\vec{a}_{cm}$$

thus

$$M\vec{a}_{CM} = \vec{F}_{ext} \tag{115}$$

Example: acrobat in the air - CM moves in a simple parabola.

If $\vec{F}_{ext} = 0$

$$\vec{V}_{CM} = const \tag{116}$$

Examples: boat on a lake, astronaut.

D. Advanced: Energy and CM

$$\vec{v}_i = \vec{v}_i' + \vec{V}_{CM}$$

Note:

$$\sum_i m_i \vec{v}_i' = 0$$

$$E = \frac{1}{2}\sum_i m_i \left(\vec{v}_i' + \vec{V}_{CM}\right)^2$$

$$E = \frac{1}{2}\sum_i m_i \left(v_i'\right)^2 + \left(\sum_i m_i \vec{v}_i' \cdot \vec{V}_{CM}\right) + \frac{1}{2}MV_{CM}^2$$

$$E = \frac{1}{2}\sum_i m_i \left(v_i'\right)^2 + \frac{1}{2}MV_{CM}^2$$

2nd term - KE of the CM, 1st term - KE *relative* to CM. (will be very important for rotation).

XIII. COLLISIONS

Very large forces F acting over very short times Δt, with a finite impulse $F_a\,\Delta t$. Impulse of "regular" forces (e.g. gravity) over such tiny time intervals is negligible, so the colliding bodies are almost an insolated system with

$$\vec{\mathcal{P}} \simeq const$$

before and after the collision. With energies it is not so simple, and several options are possible.

A. Inelastic

$$\vec{\mathcal{P}} = const\ , \quad K \neq const \tag{117}$$

1. *Perfectly inelastic*

After collision:

$$\vec{V_1} = \vec{V_2} \tag{118}$$

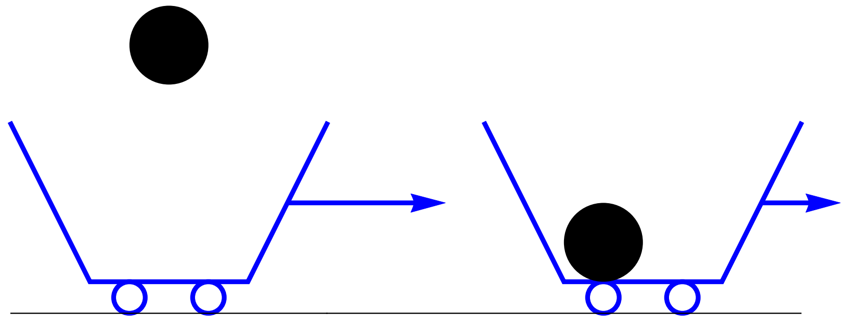


FIG. 18: Example (1D). A chunk of black coal of mass m falls vertically into a vagon with mass M, originally movin with velocity V. (Only) the horizontal component of momentum is conserved: $m \cdot 0 + M \cdot V = (M + m)v_{final}$

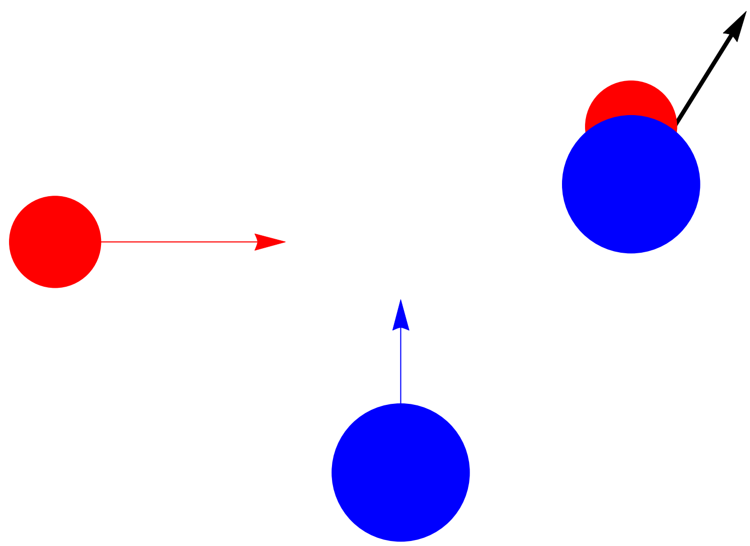


FIG. 19:

Example (2D). A car m (red) moving East with velocity v collides and hooks up with a truck M moving North with velocity V. Find the magnitude and direction of the resulting velocity V_1. Solution:

Direction:

$$\tan\theta = \frac{V_{1y}}{V_{1x}} = \frac{P_y}{P_x} = \frac{MV}{mv}$$

Magnitude:

$$(M + m)\vec{V_1} = m\vec{v} + M\vec{V}$$

$$V_{1x} = mv/(M + m)\,,\ V_{1y} = MV/(M + m)$$

$$V_1 = \sqrt{V_{1x}^2 + V_{1y}^2}$$

Note: in inelastic collision energy is always lost:

$$\frac{1}{2}(M+m)V^2 - \left(\frac{1}{2}mv_1^2 + \frac{1}{2}Mv_2^2\right) < 0$$

2. Explosion

A rocket with $M, \vec{v}$ brakes into m_1, $\vec{V}_1$ and m_2, $\vec{V}_2$ with $m_1 + m_2 = M$.

$$M\vec{v} = m_1\vec{V}_1 + m_2\vec{V}_2$$

Energy is increased.

B. Elastic

$$\vec{\mathcal{P}} = const\ , \quad K = const \tag{119}$$

1. Advanced: 1D collison, $m \neq M$

Collision of a body $m\,, v$ with a stationary body M.
From conservation of momentum:

$$m\,(v - V_1) = MV_2$$

From conservation of energy:

$$\frac{m}{2}\left(v^2 - V_1^2\right) = \frac{M}{2}V_2^2$$

Divide 2nd equation by the 1st one $(V_2 \neq 0\ !)$ to get

$$v + V_1 = V_2$$

Use the above to replace V_2 in equation for momentum to get

$$V_1 = \frac{m - M}{m + M}v$$

and then

$$V_2 = \frac{2m}{M+m} v$$

Note limits and special cases:

- $m = M$: $V_1 = 0$, $V_2 = v$ (”total exchange”)
- $M \to \infty$ (collision with a wall): $V_1 = -v$ (reflection)
- m (”tennis racket”) $\gg$ M (”tennis ball”): $V_2 = 2v$

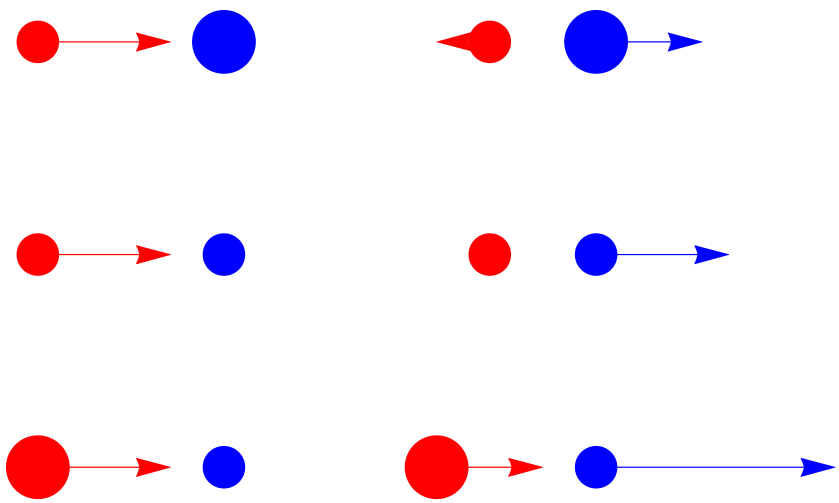


FIG. 20: Elastic collision of a moving missile particle m (red) with a stationary target M. (a) $m < M$; (b) $m = M$; (c) $m > M$. In each case both energy and momentum are conserved.

2. *Advanced: 2D elastic colision of two isentical masses*

Collisions of 2 identical billiard balls (2nd originally not moving).

$$\vec{\mathcal{P}}\ (before\ collision) = \vec{\mathcal{P}}\ (after)$$

$$K\ (before\ collision) = K\ (after)$$

Momentum conservation gives

$$\vec{v} = \vec{V}_1 + \vec{V}_2$$

Energy conservation gives

$$v^2/2 = V_1^2/2 + V_2^2/2$$

or

$$\vec{V}_1 \cdot \vec{V}_2 = 0$$

which is a 90^o angle for any $V_2 \neq 0$.

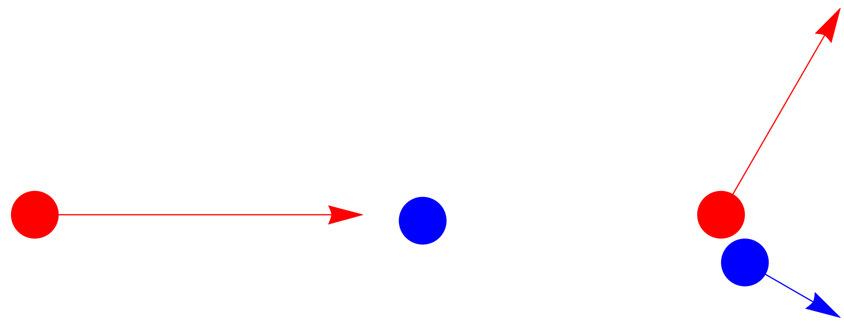


FIG. 21: Off-center elastic collision of 2 identical billiard balls (one stationary). Both energy and momentum are conserved. Note that the angle between final velocities is always 90^o.

Advanced Collisions of elementary particles

$$K = \frac{p^2}{2m} \tag{120}$$

only for $v \ll c$. General

$$E = \sqrt{m^2c^4 + c^2p^2}$$

Two limits:

$$m = 0\,,\ E = cp$$

("photon", $v = c$) and

$$m \neq 0\,,\ v \ll c : E \approx mc^2 + \frac{p^2}{2m}$$

XIV. KINEMATICS OF ROTATION

A. Radian measure of an angle

see Fig. 22. Arc length

$$l = r\theta \tag{121}$$

if θ is measured in radians.

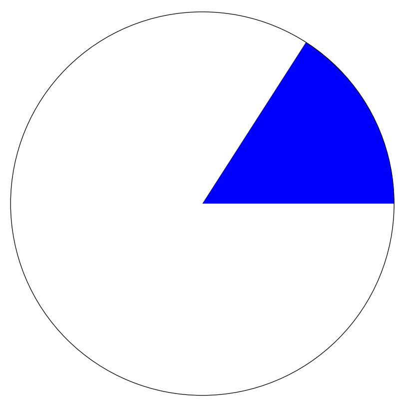


FIG. 22: Angle of $1\,rad \approx 57.3^o$. For this angle the length of the circular arc exactly equals the radius. The full angle, 360^o, is 2π radians.

B. Angular velocity

Notations: ω (omega)

Units: rad/s

Definition:

$$\omega = \frac{d\theta}{dt} \approx \frac{\Delta\theta}{\Delta t} \tag{122}$$

Conversion from revolution frequency ($\omega = const$):

Example: find ω for $45\, rev/min$

$$45\,\frac{rev}{min} = 45\,\frac{2\pi\, rad}{60\, s} \approx 4.7\frac{rad}{s}$$

C. Connection with linear velocity and centripetal acceleration for circular motion

$$v = \frac{dl}{dt} = \frac{d(r\theta)}{dt} = \omega r \tag{123}$$

$$a_c = v^2/r = \omega^2 r \tag{124}$$

Advanced. $\vec{\omega}$ as a vector:

Direction - along the axis of revolution (right-hand rule). Relation to linear velocity

$$\vec{v} = \vec{\omega} \times \vec{r}$$

D. Angular acceleration

Definition:

$$\boxed{\alpha = \frac{d\omega}{dt} = \frac{d^2\theta}{dt^2}} \tag{125}$$

Units:

$$[\alpha] = rad/s^2$$

E. Connection with tangential acceleration

$$a_\tau = \frac{dv}{dt} = r\frac{d\omega}{dt} = r\alpha \tag{126}$$

(very important for rolling problems!)

F. Rotation with $\alpha = const$

Direct analogy with linear motion:

$$x \;\; \rightarrow \; \theta \;\; , \quad v \; \rightarrow \; \omega \;\; , \quad a \; \rightarrow \; \alpha$$

$$\omega = \omega_0 + \alpha t \tag{127}$$

$$\theta = \theta_0 + \omega_0 t + \frac{1}{2}\alpha t^2 \tag{128}$$

$$\theta - \theta_0 = \frac{\omega^2 - \omega_0^2}{2\alpha} \tag{129}$$

New: connection between θ (in rads) and N (in revs) and ω in rad/s and frequency $f = 1/T$ in rev/s:

$$\boxed{\theta = 2\pi N} \; , \quad \boxed{\omega = 2\pi f} \tag{130}$$

Examples:

- a free spinning wheel makes $N = 1000$ revolutions in $t = 10$ seconds, and stops. Find α. *Solution*: in formulas

$$\omega = 0\,,\ \theta - \theta_0 = 2\pi N$$

or

$$0 = \omega_0 + \alpha t\,,\ 2\pi N = \omega_0 t + \frac{1}{2}\alpha t^2$$

Thus,

$$\omega_0 = -\alpha t\,,\ 2\pi N = (-\alpha t)t + \frac{1}{2}t^2$$

and

$$2\pi N = -\frac{1}{2}\alpha t^2\,,\ \alpha = -\frac{4\pi N}{t^2} = \ldots$$

- Given $\omega(0) = 5\,rad/s\,,\ t = 1\,s\,,\ N = 100\,rev$ (does not stop!). Find α.

$$2\pi N = \omega(0)t + \frac{1}{2}\alpha t^2\,,\ \alpha = 2\frac{2\pi N - \omega(0)t}{t^2} = \ldots$$

XV. KINETIC ENERGY OF ROTATION AND ROTATIONAL INERTIA

A. The formula $K = 1/2\, I\omega^2$

For any point mass

$$K_i = \frac{1}{2}m_i v_i^2 \tag{131}$$

For a solid rotating about an axis

$$v_i = \omega r_i \tag{132}$$

with r_i being the distance from the axis and ω, the angular velocity being *the same* for every point. Thus, the full kinetic energy is

$$K = \sum_i K_i = \frac{1}{2}\omega^2 \sum_i m_i r_i^2 \equiv \frac{1}{2}I\omega^2 \tag{133}$$

Here I, the rotational inertia, is the property of a body, independent of ω (but sensitive to selection of the rotational axis):

$$\boxed{I = \sum_i m_i r_i^2}\ , \quad \boxed{K = \frac{1}{2} I \omega^2} \tag{134}$$

Or, for continuous distribution of masses:

$$I = \int dl\, \lambda r^2 \tag{135}$$

for a linear object with linear density λ (in kg/m); or

$$I = \int dS\, \sigma r^2 \tag{136}$$

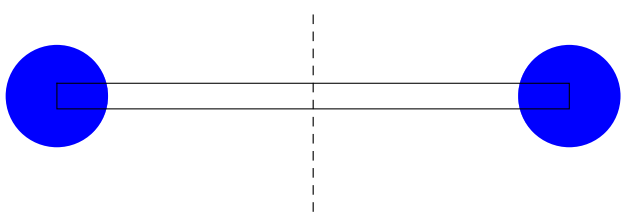
for a flat object (S-area) with planar density σ (in kg/m^2), or

$$I = \int dV\, \rho r^2 \tag{137}$$

for a 3D object (V-volume) with density ρ (in kg/m^3)

B. Rotational Inertia: Examples

1. *Collection of point masses*



Two identical masses m at $x = \pm a/2$. Rotation in the xy plane about the z-axis through the CM.

$$I = 2 \cdot m(a/2)^2 = ma^2/2 = \frac{1}{4} M a^2\ , \quad M = m + m \tag{138}$$

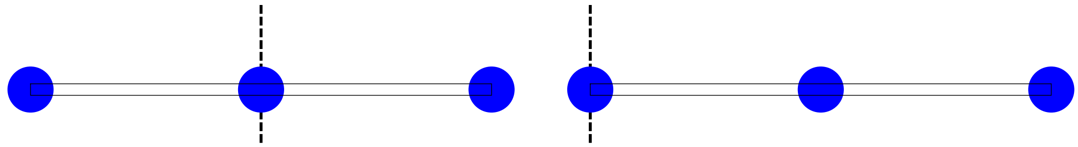


FIG. 23: 3 identical point masses m on a massless rod of length L. Left: axis through CM. $I_{CM} = m(L/2)^2 + m \cdot 0 + m(L/2)^2 = mL^2/2$. Right: axis through the end. $I_{end} = m \cdot 0 + m(L/2)^2 + mL^2 = (5/4)mL^2 > I_{CM}$ (!).

2. Hoop

Hoop of mass M, radius R in the xy plane, center at the origin. Rotation in the xy plane about the z-axis.

Linear density

$$\lambda = M/(2\pi R)$$

$$I_{hoop} = \int_0^{2\pi R} dl\, \lambda R^2 = MR^2 \tag{139}$$

(the same for hollow cylinder about the axis)

3. Rod

Uniform rod of mass M between at $x = \pm l/2$. Rotation in the xy plane about the z-axis through the center of mass.

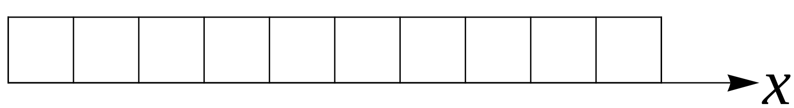


FIG. 24: Evaluating rotational inertia of a rod

Linear density

$$\lambda = M/l$$

Thus,

$$I = \int_{-l/2}^{l/2} dx\, \lambda x^2 = 2\lambda \cdot \int_0^{l/2} dx\, x^2 = 2\lambda \frac{1}{3}(l/2)^3$$

Or,

$$I_{rod} = \frac{Ml^2}{12} \tag{140}$$

4. Disk

Uniform disk of mass M, radius R in the xy plane, center at the origin. Rotation in the xy plane about the z-axis.

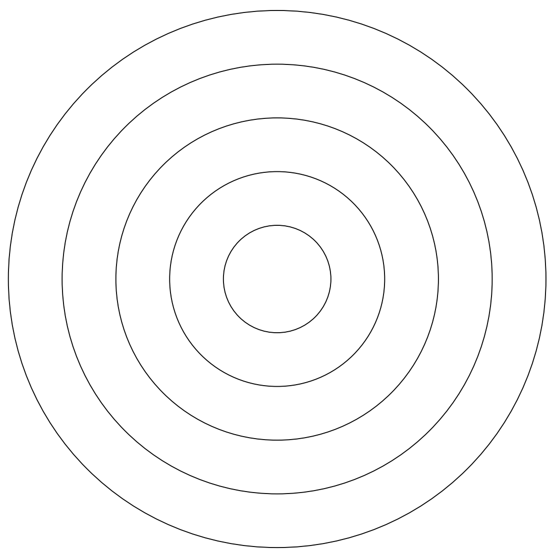


FIG. 25: Evaluating rotational inertia of a disk

Planar density

$$\sigma = M/\left[\pi R^2\right]$$

Elementary area

$$dS = 2\pi r\, dr$$

$$I_{disk} = \int_0^R dr\, 2\pi r \sigma r^2 = \frac{1}{2} M R^2 \tag{141}$$

(the same for solid cylinder about the axis).

5. *Advanced: Solid and hollow spheres*

Solid sphere: slice it into collection of thin disks of thickness dz and radius $r = \sqrt{R^2 - z^2}$

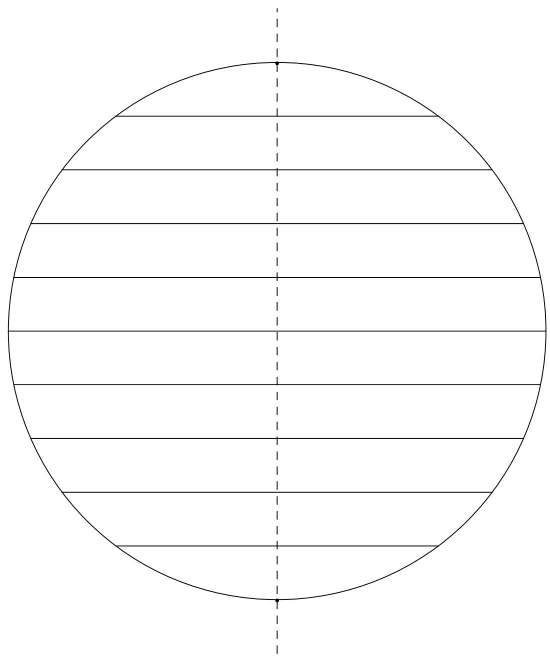


FIG. 26: Evaluating rotational inertia of a solid sphere

$$I_{sph} = \int_{-R}^R dz\, \frac{1}{2} \pi r^2 \rho r^2 = \frac{2}{5} M R^2 \tag{142}$$

Hollow:

$$I_{h.sph} = \frac{2}{3} M R^2$$

C. Parallel axis theorem

$$\boxed{I = I_{cm} + MD^2} \tag{143}$$

with I_{cm} being rotational inertia about a parallel axis passing through the center of mass and D - distance to that axis.

Proof:

$$\vec{R}_{cm} = \frac{1}{M}\sum_i \vec{r}_i m_i$$

Introduce

$$\vec{r}_i' = \vec{r}_i - \vec{R}_{cm}$$

with

$$\sum_i \vec{r}_i' m_i = 0$$

and

$$I_{cm} = \sum_i m_i \left(r_i'\right)^2$$

Now

$$I = \sum_i m_i \left(\vec{r}_i' + \vec{D}\right)^2 = I_{cm} + MD^2 + 2\vec{D}\cdot\sum_i \vec{r}_i' m_i$$

where the last sum is zero, which completes the proof.

Example: rod M, L about the end

$$I_{CM} = ML^2/12$$

with $D = L/2$:

$$I_{rod,end} = ML^2/12 + M(L/2)^2 = ML^2/3$$

1. Distributed bodies plus point masses

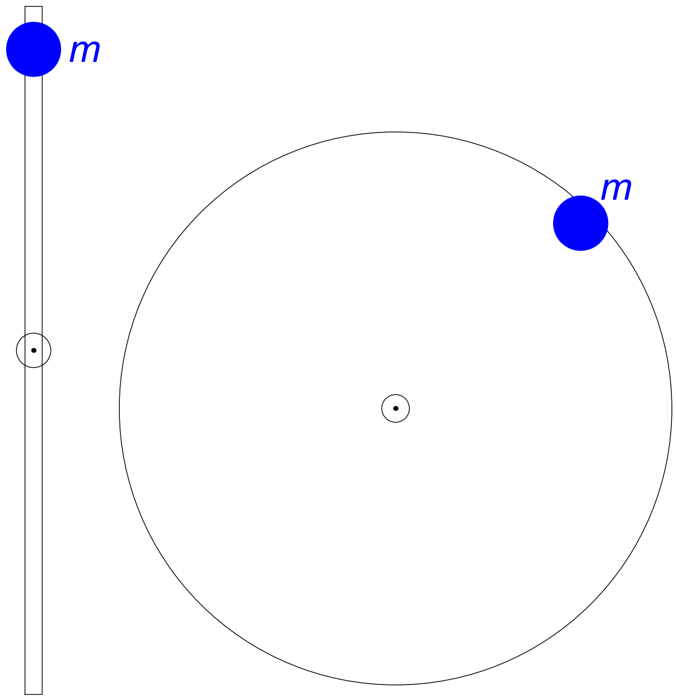
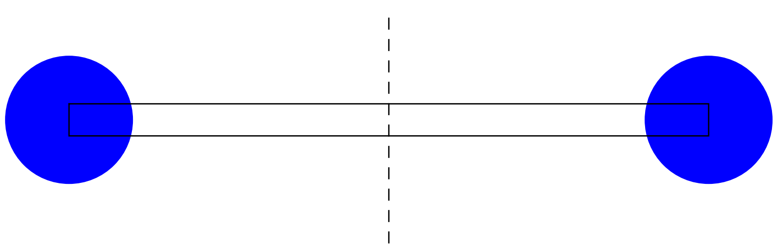


FIG. 27: Modification of I by a point mass m added at distance a from center. Left: rod mass M, length L. Right: disk of mass M and radius R.

$$rod: \; I = I_{rod} + ma^2 = ML^2/12 + ma^2 = L^2(M/3 + m)/4 \;\text{ if } a \simeq L/2$$

$$disk: \; I = I_{disk} + ma^2 = MR^2/2 + ma^2 = R^2(M/2 + m) \;\; if \; a \simeq R$$

2. Advanced: Combinations of distributed bodies.



Example: Solid spheres with radius a and mass M, rod length L; rotation axis through center.

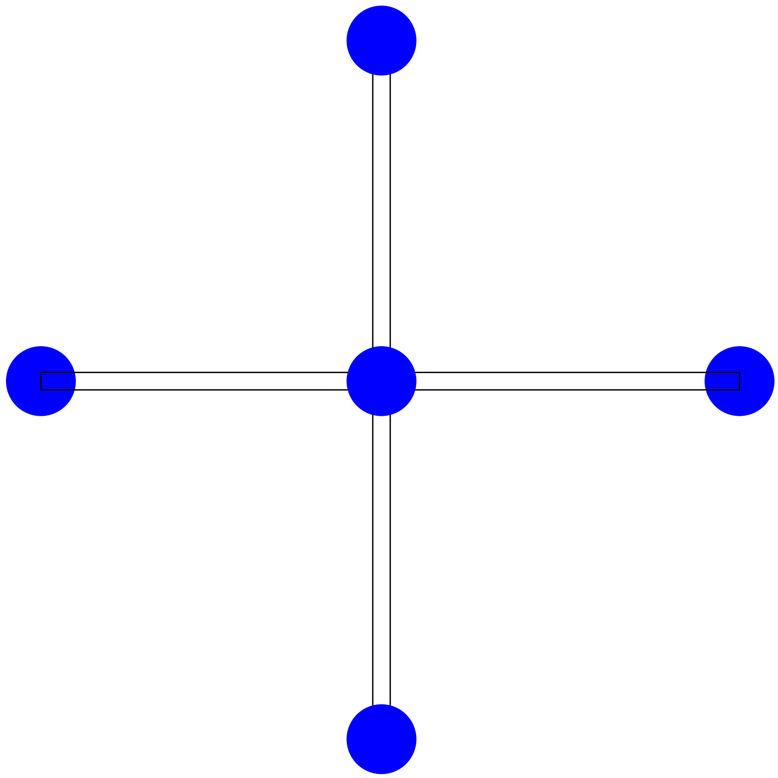
1) negligible size of spheres ($a \ll L$) and massless rod

$$I = 2 \times M(L/2)^2 = ML^2/2$$

2) non-negligible size of spheres

$$I = 2 \times \frac{2}{5}Ma^2 + ML^2/2$$

3) rod of mass m - add $mL^2/12$



Example:

Solid spheres with radius a and mass M each, rods with length L each; rotation axis through center, perpendicular to plane.

1) negligible size of spheres ($a \ll L$) and massless rods

$$I = 4 \times M(L/2)^2 = ML^2$$

Note: central mass does not contribute!

2) non-negligible size of spheres

$$I = 5 \times \frac{2}{5}Ma^2 + ML^2$$

Note: all 5 spheres contribute.

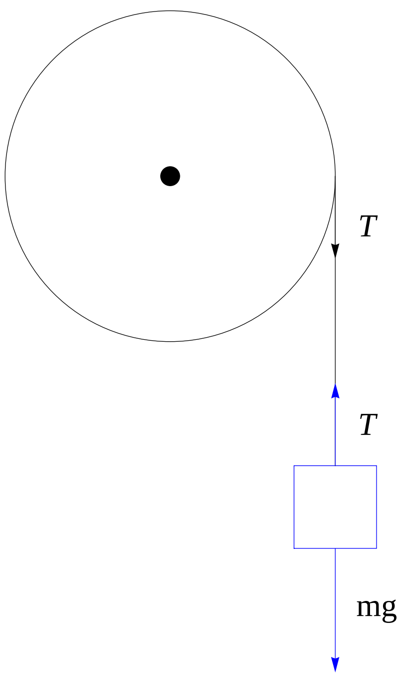
3) rods of mass m each - add $2 \times mL^2/12$

D. Conservation of energy, including rotation

$$\boxed{K + U = const} \qquad (144)$$

where K is the *total* kinetic energy (translational and rotational for all bodies) and U is total potential energy.

E. "Bucket falling into a well"



Suppose mass m goes distance h down starting from rest. Find final velocity v. Pulley is a disk M, R. Solution: ignore T (!), use energy only.

- energy conservation

$$\frac{1}{2}mv^2 + \frac{1}{2}I\omega^2 = mgh$$

• constrains

$$v = \omega R$$

Thus

$$v^2 = 2gh \frac{m}{m + I/R^2}$$

If $I = MR^2/2$ (disk)

$$v^2 = 2gh \frac{1}{1 + M/(2m)}\ , \quad v = \ldots$$

1. Advanced: Atwood machine

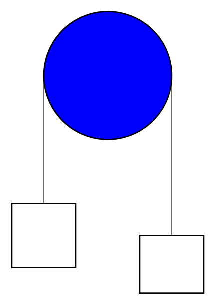


FIG. 28: Atwood machine. Mass M (left) is almost balanced by a slightly smaller mass m. Pulley has rotational inertia I and radius R.

Suppose the larger mass goes distance h down starting from rest. Find final velocity v.

• energy conservation

$$\frac{1}{2}(M + m)v^2 + \frac{1}{2}I\omega^2 = (M - m)gh$$

• constrains

$$v = \omega R$$

Thus

$$v^2 = 2gh\frac{M-m}{M+m+I/R^2}$$

Acceleration from

$$h = v^2/2a$$

which gives

$$a = g\frac{M-m}{M+m+I/R^2}$$

Note the limit: $m = 0$, $I = 0$ gives $a = g$ (free fall).

2. Rolling

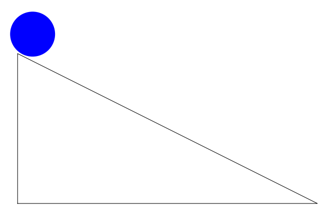


FIG. 29: Rolling down of a body with mass m, rotational inertia I and radius R. Potential energy at the top, mgh equals the full kinetic energy at the bottom, $mv^2/2 + I\omega^2/2$.

Suppose the body rolls vertical distance $h = L\sin\theta$ starting from rest. Find final velocity v.

- energy conservation

$$\frac{1}{2}mv^2 + \frac{1}{2}I\omega^2 = mgh$$

- constrains

$$v = \omega R$$

Thus

$$v^2 = 2gh\frac{1}{1+I/(mR^2)}$$

$$hoop:\ I = MR^2\ , \quad v^2 = 2gh\frac{1}{1+1}$$

$$disk:\ I = MR^2/2\ , \quad v^2 = 2gh\frac{1}{1+1/2}$$

The disk wins!

Advanced. Acceleration from

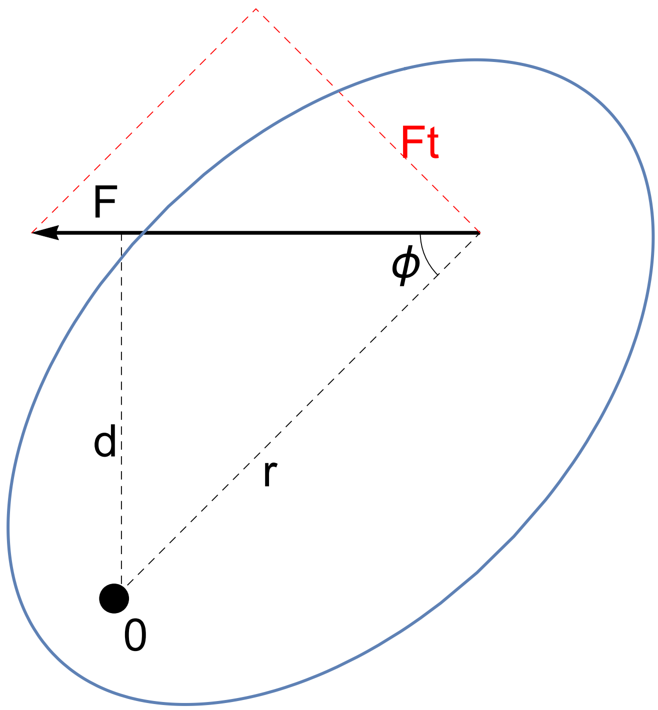
$$a = v^2/2l\ ,\ l = h/\sin\theta$$

which gives

$$a = g\sin\theta\frac{1}{1+I/\left(mR^2\right)}$$

Note the limit: $I = 0$ gives $a = g\sin\theta$.

XVI. TORQUE



A. Definition

Consider a point mass m at a fixed distance r from the axis of rotation. Only motion in tangential direction is possible. Let F_t be the tangential component of force. The *torque* is defined as

$$\boxed{\tau = F_t r = F r \sin\phi = F d} \tag{145}$$

with ϕ being the angle between the force and the radial direction. $r\sin\phi = d$ is the "lever arm". Counterclockwise torque is positive, and for several forces torques add up.

Units:

$$[\tau] = N \cdot m$$

(same as *Joules*).

B. 2nd Law for rotation

Start with a single point mass. Consider the tangential projection of the 2nd Law

$$F_t = ma_t$$

Now multiply both sides by r and use $a_t = \alpha r$ with α the angular acceleration.

$$\tau = mr^2\alpha$$

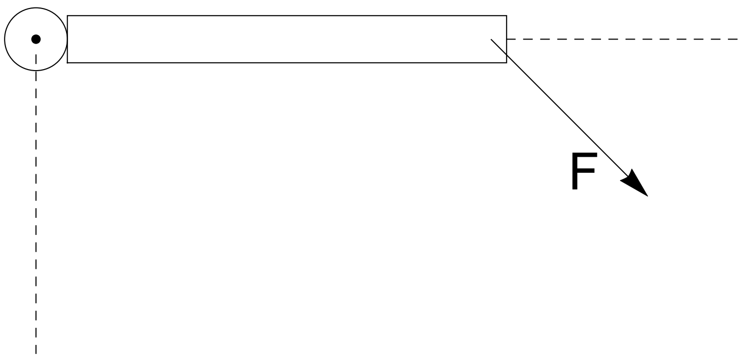
For a system of particles m_i each at a distance r_i and the same α

$$\boxed{\sum \tau = I\alpha} \tag{146}$$

C. Application of $\tau = I\alpha$

1. Revolving door

How long will it take to open a heavy, freely revolving door by 90 degrees starting from rest, if a constant force F is applied at a distance r away from the hinges at an angle ϕ, as shown in the figur? Make some reasonable approximations about parameters of the door, F and r. For I you can use $1/3\, ML^2$ ("rod"), with L being the horizontal dimension.



Solution: 1) find torque; 2) use 2nd law for rotation to find α; 3) use kinematics to estimate t

1.

$$\tau = Fr\sin\phi$$

2.

$$\alpha = \tau/I = 3Fr\sin\phi/(ML^2)$$

if $r = L$

$$\alpha = 3F/(ML)\sin\phi$$

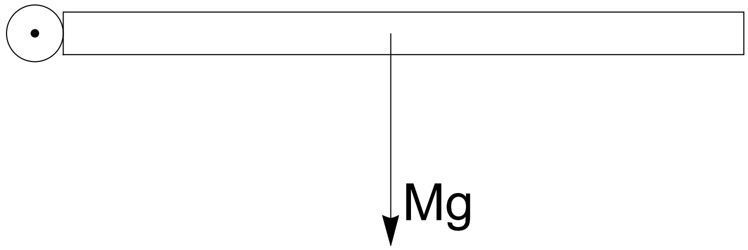
3. To maximize α use $\phi = \pi/2$. From

$$\theta = 1/2\,\alpha t^2$$

$$t = \sqrt{2\theta/\alpha} = \sqrt{\pi\frac{I}{FL}}$$

Using, e.g. $M = 30\,kg$, $F = 30\,N$, $L = 1\,m$ one gets $t \sim 1\,s$, which is reasonable.

2. Rotating rod

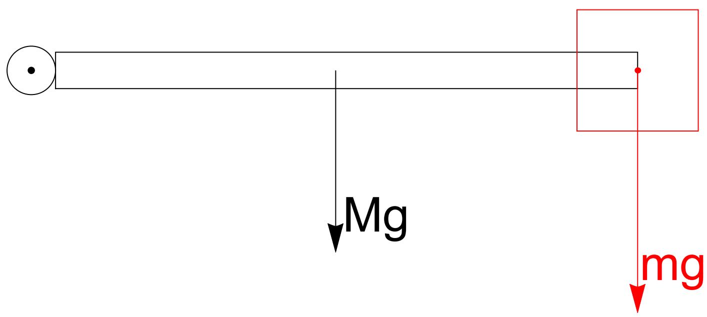


$$\alpha = \tau/I = (1/2\,Mgl) \Big/ \left(\frac{1}{3}Ml^2\right) = \frac{3}{2}g/l$$

Linear acceleration of the end:

$$a = \alpha l = \frac{3}{2}g \; > \; g \; (!)$$

3. Rotating rod with a point mass m at the end.



Will it go faster or slower?

Solution:

same as above, but

$$I \;\rightarrow\; 1/3\,Ml^2 + ml^2\,, \;\; \tau \;\rightarrow\; 1/2\,Mgl + mgl$$

Thus,

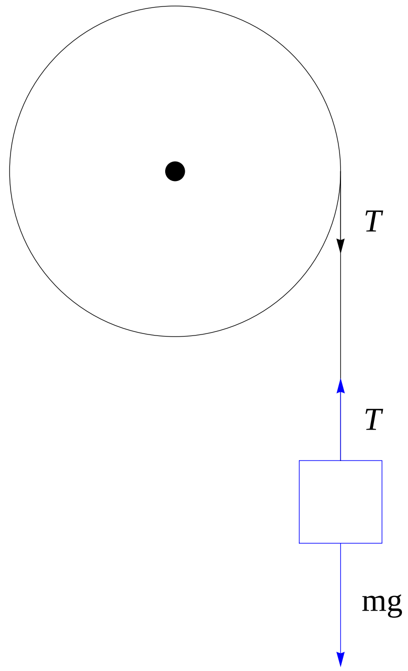
$$\alpha = \frac{3}{2}\frac{g}{l}\,\frac{1 + 2m/M}{1 + 3m/M}$$

Linear acceleration of the end:

$$\alpha = \frac{3}{2}g\frac{1 + 2m/M}{1 + 3m/M}$$

(which is smaller than before, but still larger than g)

4. "Bucket falling into a well" revisited.



2nd Law for rotation:

$$\alpha = \tau/I = TR/I$$

2nd Law for linear motion:

$$ma = mg - T$$

Constrain:

$$\alpha = a/R$$

Thus (divide 1st equation by R/I and add to the 2nd one, replacing α):

$$aI/R^2 + ma = mg$$

or

$$a = g\frac{1}{1 + I/(mR^2)}$$

5. Advanced: Atwood machine revisited.

Let T_1 and T_2 be tensions in left string (connected to larger mass M) and in the right string, respectively.

- 2nd Law(s) for each body (and for the pulley with $\tau = (T_1 - T_2)\, R$)

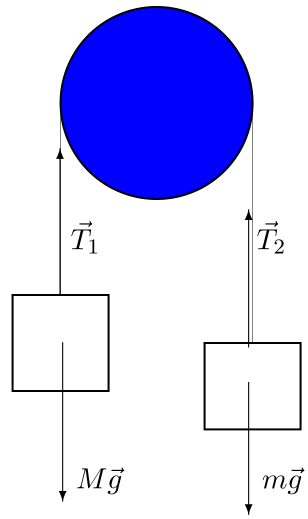


FIG. 30: Atwood machine. Mass M (left) is almost balanced by a slightly smaller mass m. Pulley has rotational inertia I and radius R.

- constrains $a = \alpha R$

From 2nd Law(s):

$$Mg - T_1 = Ma\,,\ T_2 - mg = ma\,,\ (T_1 - T_2) = I\alpha/R$$

Add all 3 together to get (with constrains)

$$(M - m)g = (M + m)a + I\alpha/R = (M + m + I/R^2)a$$

which gives

$$a = g\frac{M - m}{M + m + I/R^2}$$

and $\alpha = a/R$. What if need tension?

$$T_1 = Mg - ma\ <\ Mg\ ,\ \ T_2 = mg + ma\ >\ mg$$

6. Rolling down incline revisited.

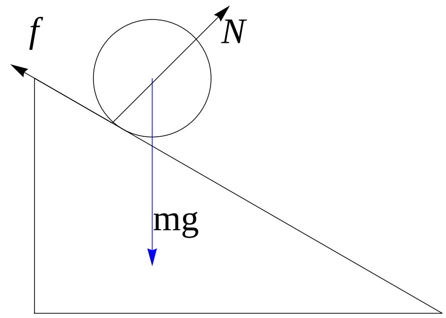


FIG. 31: Rolling down of a body with mass m, rotational inertia I and radius R. Three forces act on the body: $\vec{f}$ - static friction at the point of contact, up the plane; $\vec{N}$ - normal reaction, perpendicular to the plain at the point of contact and $m\vec{g}$ is applied to CM. Note that only friction has a torque with respect to CM.

- 2nd Law(s) for linear and for rotational accelerations with torque $\tau = fR$ (f - static friction)
- constrains $a = \alpha R$

2nd Law (linear)

$$\vec{f} + \vec{N} + m\vec{g} = m\vec{a}$$

or with x-axis down the incline

$$-f + mg\sin\theta = ma$$

2nd Law (rotation)

$$fR = I\alpha$$

or with constrain

$$f = aI/R^2$$

Thus,

$$-aI/R^2 + mg\sin\theta = ma$$

or

$$a = g\sin\theta\,\frac{1}{1 + I/mR^2}$$

the same as from energy conservation.

Advanced. Alternative solution: "Rotation" about point of contact with $I' = I + mR^2$ ("parallel axes theorem"). Now only gravity has torque $\tau = mgR\sin\theta$. Thus

$$\alpha = \tau/I' = \frac{g}{R}\sin\theta\,\frac{1}{1 + I/mR^2}$$

which gives the same $a = \alpha R$.

D. Torque as a vector

1. Cross product

see Introduction on vectors

2. Vector torque

$$\vec{\tau} = \vec{r} \times \vec{F} \tag{147}$$

XVII. ANGULAR MOMENTUM $\mathcal{L}$

A. Single point mass

$$\boxed{\vec{\mathcal{L}} = \vec{r} \times \vec{p}} \tag{148}$$

with $\vec{p} = m\vec{v}$, the momentum.

Example. Find $\vec{\mathcal{L}}$ for circular motion.

Solution: Direction - along the axis of rotation (as $\vec{\omega}$!). Magnitude:

$$\mathcal{L} = mvr\sin 90^o = mr^2\omega$$

or

$$\vec{\mathcal{L}} = mr^2\vec{\omega} \tag{149}$$

B. System of particles

$$\vec{\mathcal{L}} = \sum_i \vec{r}_i \times \vec{p}_i \tag{150}$$

C. Rotating symmetric solid

1. Angular velocity as a vector

Direct $\vec{\omega}$ allong the axis of rotation using the right-hand rule.

Example. Find $\vec{\omega}$ for the spinning Earth.

Solution: Direction - from South to North pole. Magnitude:

$$\omega = \frac{2\pi\, rad}{24 \cdot 3600\, s} \simeq \ldots \frac{rad}{s}$$

If axis of rotation is also an axis of symmetry for the body

$$\boxed{\mathcal{L} = \sum_i m_i r_i^2 \omega = I\omega} \tag{151}$$

or

$$\vec{\mathcal{L}} = I\vec{\omega}$$

D. 2nd Law for rotation in terms of $\vec{\mathcal{L}}$

Start with

$$\vec{F} = \frac{d\vec{p}}{dt}$$

Then

$$\vec{\tau} = \frac{d\vec{\mathcal{L}}}{dt} \tag{152}$$

XVIII. CONSERVATION OF ANGULAR MOMENTUM

Start with

$$\vec{\tau} = \frac{d\vec{\mathcal{L}}}{dt}$$

If $\vec{\tau} = 0$ (no net external torque)

$$\vec{\mathcal{L}} = const \tag{153}$$

valid everywhere (from molecules and below, to stars and beyond).

For a closed *mechanical* system, thus

$$E = const\,,\ \vec{P} = const\,,\ \vec{\mathcal{L}} = const$$

For *any* closed system (with friction, inelastic collisions, break up of material, chemical or nuclear reactions, etc.)

$$E \neq const\,,\ \vec{P} = const\,,\ \vec{\mathcal{L}} = const$$

A. Examples

1. Free particle

$$\vec{\mathcal{L}}(t) = \vec{r}(t) \times m\vec{v}$$

with $\vec{v} = const$ and

$$\vec{r}(t) = \vec{r}_0 + \vec{v}t$$

Thus, from $\vec{v} \times \vec{v} = 0$

$$\vec{\mathcal{L}}(t) = (\vec{r}_0 + \vec{v}t) \times m\vec{v} = \vec{r}_0 \times m\vec{v} = \vec{\mathcal{L}}(0)$$

2. Student on a rotating platform

(in class demo) Let I be rotational inertia of student+platform, and

$$I' \simeq I + 2Mr^2$$

the rotational inertia of student+platform+extended arms with dumbbels (r is about the length of an arm). Then,

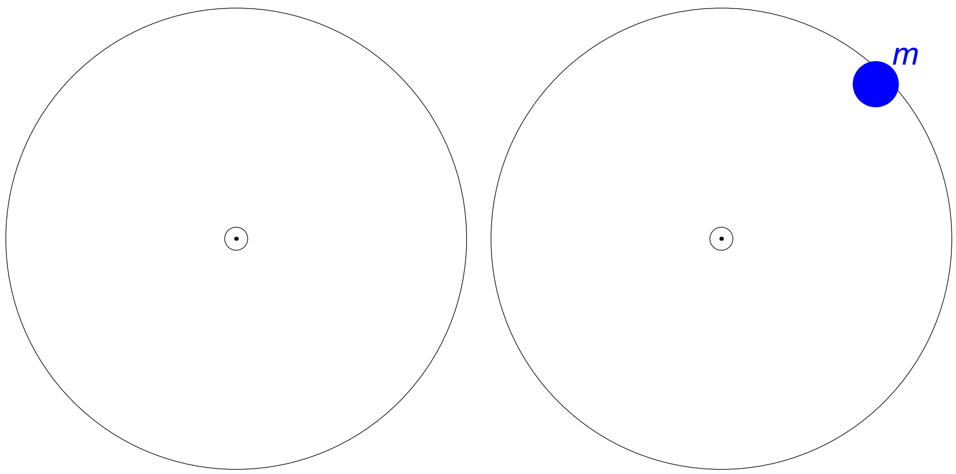
$$\mathcal{L} = I\omega \;=\; I'\omega'$$

or

$$\omega' = \omega\frac{I}{I'} = \omega\frac{1}{1+2Mr^2/I}$$

3. Chewing gum on a disk

An $m = 5\,g$ object is dropped onto a uniform disc of rotational inertia $I = 2\cdot 10^{-4}\,kg\cdot m^2$ rotating freely at 33.3 revolutions per minute. The object adheres to the surface of the disc at distance $r = 5\,cm$ from its center. What is the final angular velocity of the disc?



Solution. Similarly to previous example, from conservatioon of angular momentum

$$\mathcal{L} = I\omega \;=\; I'\omega'$$

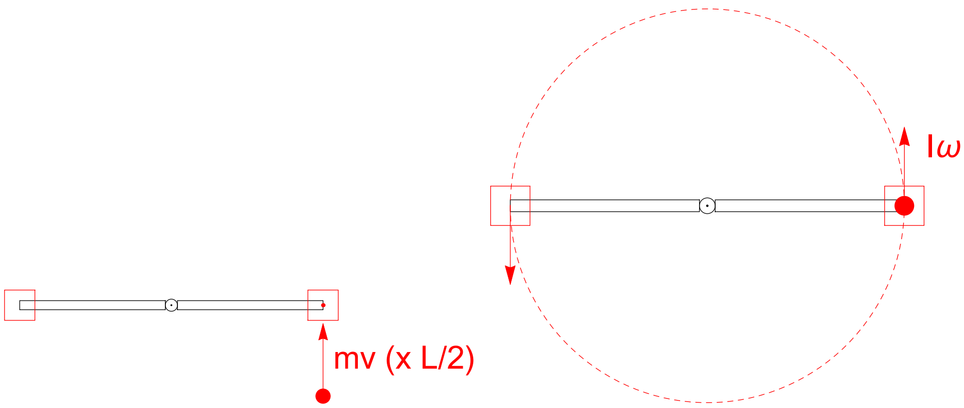
New rotational inertia

$$I' = I + mr^2$$

Thus

$$\omega' = \omega\frac{I}{I'} = \omega\frac{I}{I+mr^2} = \omega\frac{1}{1+mr^2/I}$$

To measure the speed of a fast bullet a rod (mass M_{rod}) with length L and with two wood blocks with the masses M at each end, is used. The whole system can rotate in a horizontal plane about a vertical axis through its center. The rod is at rest when a small bullet of mass m and velocity v is fired into one of the blocks. The bullet remains stuck in the block after it hits. Immediately after the collision, the whole system rotates with angular velocity ω. Find v. (use $L = 2$ meters, $\omega = 0.5\, rad/s$ and $m = 5\, g$, $M = 1\, kg$, $M_{rod} = 4\, kg$).



Solution. Let $r = L/2 = 1$ meter be the distance from the pivot. Angular momentum: before collision

$$\mathcal{L} = \frac{L}{2} mv$$

(due to bullet); after:

$$\mathcal{L} = I\omega$$

with

$$I = I_0 + m(L/2)^2$$

Here I_0 is rotational inertia without the bullet:

$$I_0 = M_{rod}L^2/12 + 2M(L/2)^2$$

Due to conservation of angular momentum

$$\frac{L}{2} mv = I\omega$$

and

$$v = 2I\omega/(mL)$$

5. Rotating star (white dwarf)

A uniform spherical star collapses to 0.3% of its former radius. If the star initially rotates with the frequency 1 rev/day what would the new rotation frequency be?

Solution - in class

XIX. EQUILIBRIUM

A. General conditions of equilibrium

$$\sum \vec{F}_i = 0 \tag{154}$$

$$\sum \vec{\tau}_i = 0 \tag{155}$$

Theorem. In equilibrium, torque can be calculated about *any* point.

Proof. Let $\vec{r}_i$ determine positions of particles in the system with respect to point O. Selecting another point as a reference is equivalent to a shift of every $\vec{r}_i$ by the same $\vec{r}_o$. Then,

$$\vec{\tau}_{new} = \sum_i (\vec{r}_i + \vec{r}_o) \times \vec{F}_i = \vec{\tau} + \vec{r}_o \times \sum_i \vec{F}_i = \vec{\tau}$$

B. Center of gravity

Theorem. For a *uniform* field $\vec{g}$ the center of gravity coinsides with the COM.

Proof. Torque due to gravity is

$$\tau_g = \sum_i \vec{r}_i \times m_i \vec{g} = \left(\sum_i m_i \vec{r}_i \right) \times \vec{g} = M\vec{R}_{CM} \times \vec{g}$$

C. Examples

1. Seesaw

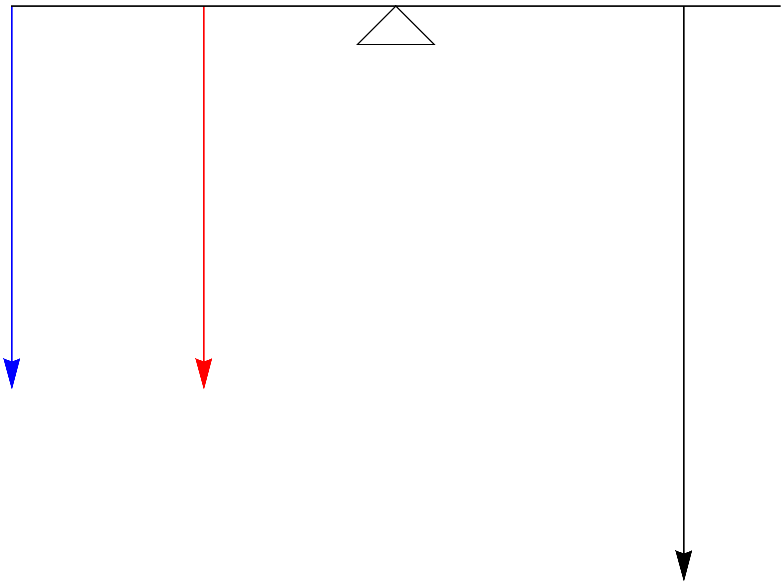


FIG. 32: Two twins, masses m and m (left) against their dad with mass M. Force of gravity on the seesaw and reaction of the fulcrum are not shown since they produce no torque.

If $2d$, d and D are distances from the fulcrum for each of the twins and the father,

$$mg \cdot 2d + mgd = MgD$$

or

$$3md = MD$$

Note that used only torque condition of equilibrium. If need reaction from the fulcrum $\vec{N}$ use the force condition

$$\vec{N} + (2m + M + M_{seesaw})\,\vec{g} = 0$$

2. Horizontal beam

Cancellation of torques gives

$$TL\sin\theta = Mg\frac{L}{2}$$

or

$$T = Mg/2\sin\theta$$

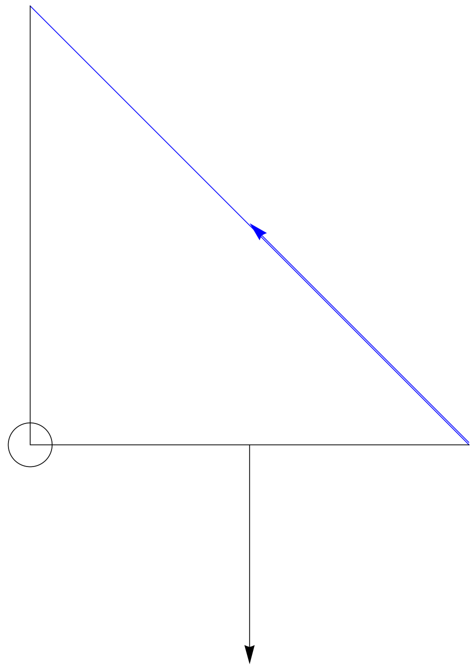


FIG. 33: Horizontal beam of mass M and length L supported by a blue cord making angle θ with horizontal. Only forces with non-zero torque about the pivot (tension $\vec{T}$ -blue- and gravity $M\vec{g}$ -black) are shown.

Note: if you try to make the cord horizontal, it will snap ($T \to \infty$). For $\theta \to \pi/2$ one has $T \to Mg/2$, as expected. The force condition will allow to find reaction from the pivot:

$$\vec{R} + M\vec{g} + \vec{T} = 0$$

(and $-\vec{R}$ will be the force on the pivot).

3. *Ladder against a wall*

Torques about the upper point:

$$NL\sin\left(\frac{\pi}{2} - \theta\right) - fL\sin\theta - Mg\frac{L}{2}\sin\left(\frac{\pi}{2} - \theta\right) = 0$$

But from force equilibrium (vertical) $N = Mg$ and (on the verge) $f = \mu N$. Thus,

$$\frac{1}{2}\cos\theta - \mu\sin\theta = 0$$

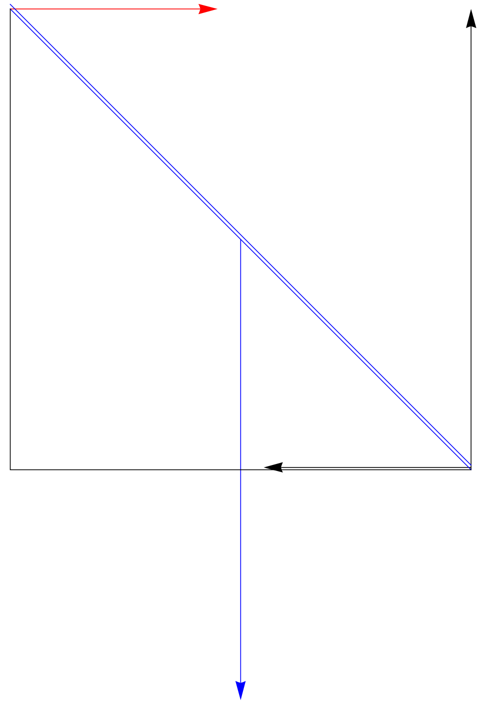


FIG. 34: Blue ladder of mass M and length L making angle θ with horizontal. Forces: $M\vec{g}$ (blue), wall reaction $\vec{F}$ (red), floor reaction $\vec{N}$ (vertical), friction $\vec{f}$ (horizontal).

XX. GRAVITATION

A. Solar system

$1\,AU \simeq 150 \cdot 10^6\,km$, about the average distance between Earth and Sun.

Mer - about $1/3\,AU$ (0.39)

V - about $3/4\,AU$ (0.73)

Mars - about $1.5\,AU$ (1.53)

J - about $5\,AU$ (5.2)

...

Solar radius - about 0.5% AU

B. Kepler's Laws

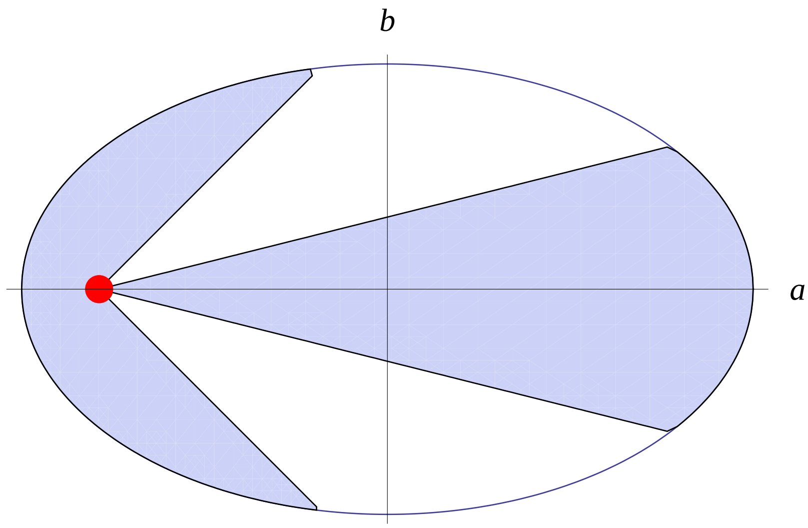


FIG. 35: The ellipse and Kepler's Law's:

a and b - major and minor semi-axes.

$f = \pm\sqrt{a^2 - b^2}$ - foci (Sun in one focus, the other one empty!)

$\epsilon = |f|/a$ - eccentricity

$a = b$ - circle

Cartesian:

$$x^2/a^2 + y^2/b^2 = 1 \tag{156}$$

Polar:

$$x = r\cos\phi\,, \; y = r\sin\phi \tag{157}$$

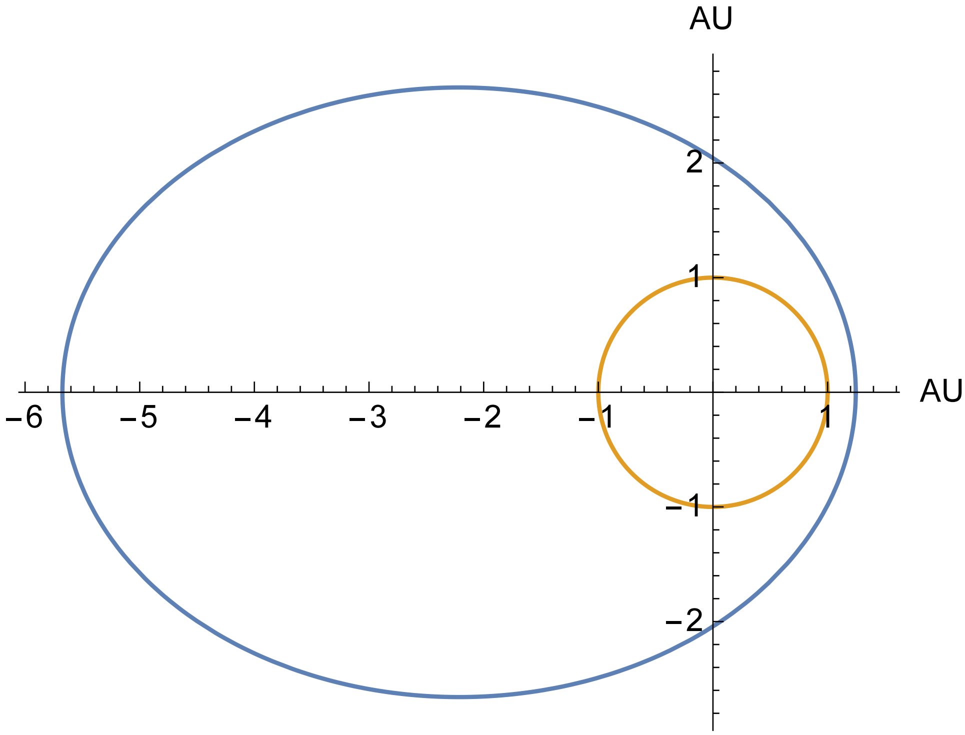
$$r(\phi) = a(1-\epsilon^2)/(1+\epsilon\cos\phi) \tag{158}$$

$\epsilon > 1$ - hyperbola, $\epsilon = 1$ - parabola ($a \to \infty$).

1. 1st law

Path of a planet - ellipse, with Sun in the focus.

(Justification is hard, and requires energy, momentum and angular momentum conservation, plus explicit gravitational force - Newton's law)



2. 2nd law

Sectorial areas during the same time intervals are same - see figure.

(Justification is easy, and requires ONLY angular momentum conservation - in class)

$$dS = \frac{1}{2}\left|\vec{r} \times \vec{v}dt\right| = \frac{1}{2m}L \cdot dt = const \cdot dt$$

3. 3rd law

$$T^2 \propto a^3 \tag{159}$$

and b does not matter!!!

(Justification is hard, and requires energy, momentum and angular momentum conservation, plus explicit gravitational force; will derive explicitly only for a circle, $a = b$, $\epsilon = 0$.)

C. The Law of Gravitation

$$\boxed{F = G\frac{Mm}{r^2}} \tag{160}$$

or in vector form

$$\vec{F} = -G\frac{Mm\vec{r}}{r^3} \tag{161}$$

with $G \approx 6.7 \cdot 10^{-11}\, N \cdot m^2/kg^2$.

1. Gravitational acceleration

$$g = F/m = G\frac{M}{r^2} = g_s\left(\frac{R}{r}\right)^2 \tag{162}$$

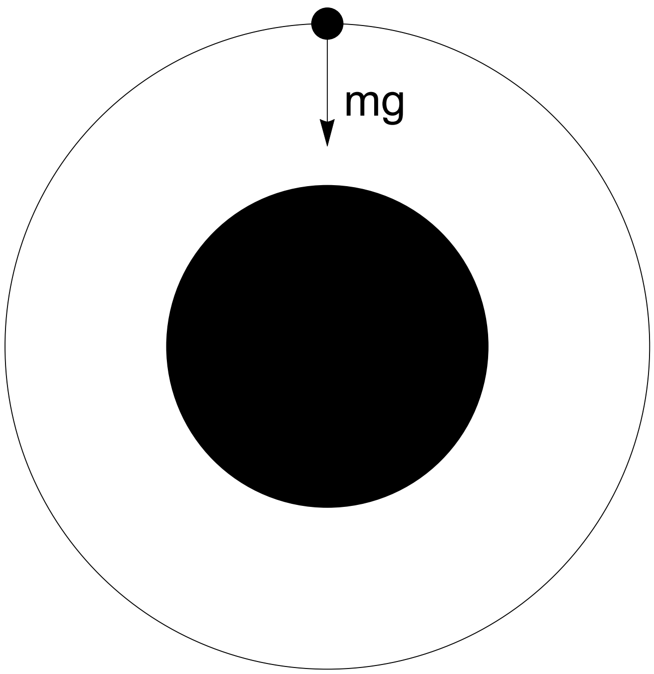
with g_s acceleration on the surface and M, R - mass and radius of the central planet (or, of the Sun). In vector form

$$\vec{g} = -G\frac{M\vec{r}}{r^3} = -\hat{r}g_s\left(\frac{R}{r}\right)^2 \;,\quad \hat{r} = \frac{\vec{r}}{r} \tag{163}$$

Example. Find g_V if $R_V \approx 0.4\,R_E$ and $M_V \approx 0.056\,M_E$

$$g \propto M/R^2 \; thus \; \frac{g_V}{g_E} = \frac{M_V}{M_E}\left(\frac{R_E}{R_V}\right)^2 \approx 0.4$$

2. Satellite



$$\frac{v^2}{r} = g(r) = GM/r^2$$

$$v = \sqrt{GM/r} \tag{164}$$

$$T = 2\pi r/v = 2\pi r^{3/2}/\sqrt{GM}$$

$$T^2 = \left(4\pi^2/GM\right) r^3 \tag{165}$$

the 3rd law. (r is distance from *center* !)

Example. Find v for earth around sun, and check if $T = 1\,year$.

$$v = \sqrt{6.7 \cdot 10^{-11}\, 2 \cdot 10^{30}/(150 \cdot 10^{9})} \approx 30\,\frac{km}{s}$$

$$T = \frac{2\pi r}{v} \approx 1\,year$$

Low orbit:

$$v^2/r = g(r) = g_s \left(\frac{R}{r}\right)^2$$

for $r \simeq R$

$$v \simeq \sqrt{g_s R} \tag{166}$$

about $8\,km/s$ for surface of Earth.

D. Energy

$$U = \int_r^\infty F(r')dr' = -GMm \int_r^\infty \frac{dr'}{(r')^2} \tag{167}$$

$$\tag{168}$$

$$\boxed{U = -G\frac{Mm}{r}} \tag{169}$$

Probe m in combined field of Sun and planet

$$U = -G\frac{M_S m}{r_{pS}} - G\frac{M_E m}{r_{pE}} \tag{170}$$

1. *Escape velocity and Black Holes*

$$E = K + U$$

$U(\infty) = 0$, $K(\infty) = E > 0$ for escape. Thus

$$\frac{1}{2}mv^2 - G\frac{Mm}{r} \geq 0\,,\; v_{esc}^2/2 = GM/R$$

$$v_{esc} = \sqrt{2GM/R} = \sqrt{2g_s R} \approx 11\,km/s$$

for the surface of Earth.

Example. A missile m is launched vertically up from the surface with $v_0 = \frac{1}{2} v_{esc}$. Find h

$$E = \frac{1}{2} m v_0^2 + \left(-G \frac{Mm}{R} \right) = \frac{1}{2} m (v_0^2 - v_{esc}^2)$$

$$E = -G \frac{Mm}{R+h} = -G \frac{Mm}{R} \frac{R}{R+h} = -\frac{1}{2} m v_{esc}^2 \frac{R}{R+h}$$

$$v_{esc}^2 \frac{R}{R+h} = v_{esc}^2 - v_0^2 = \frac{3}{4} v_{esc}^2$$

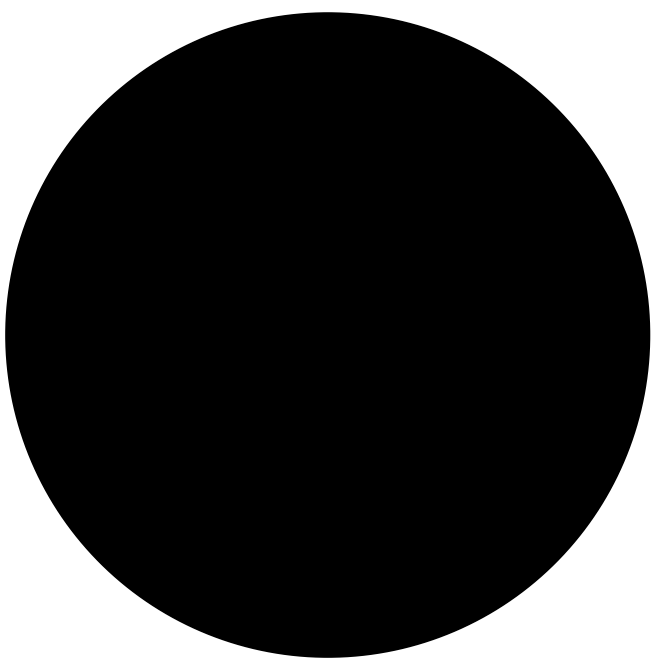
$$\frac{R}{R+h} = \frac{3}{4}, \ 1 + \frac{h}{R} = \frac{4}{3}, \ h = R/3$$

$v_{esc} = c \simeq 3 \cdot 10^5 \, km/s$ - "Black Hole". Radius of a Black Hole:

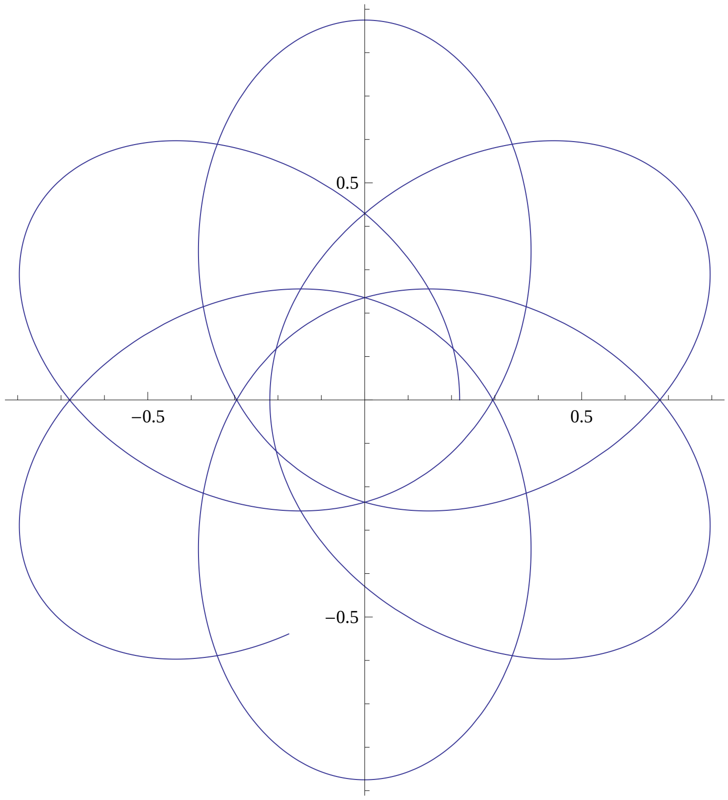
$$R_B = \frac{2GM}{c^2}$$

For Sun:

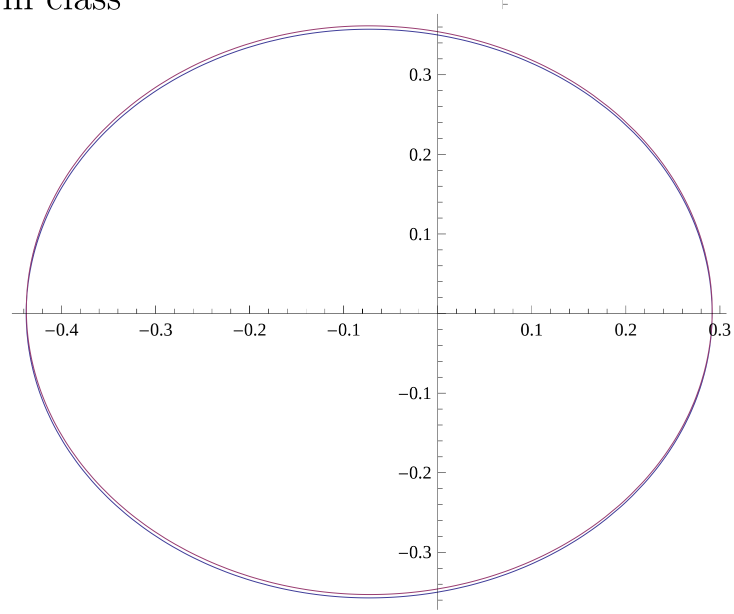
$$R_B \approx 9 \, km$$



E. Advanced: Deviations from Kepler's and Newton's laws



in class



(NOTE: Lecture on fluids is in a separate file 111_fluids.pdf)

XXI. OSCILLATIONS

A. Introduction: Math

1. $\sin(x)$, $\cos(x)$ *for small* x

$$\sin x \simeq x \tag{171}$$

error is about $-x^3/6$ and can be neglected for $x \ll 1$ [we will need this for a pendulum]

$$\cos x \simeq 1 - \frac{x^2}{2} \tag{172}$$

error is tiny for small x, about $x^4/24$.

2. *Differential equation* $\ddot{x} + x = 0$

The equation

$$\ddot{x}(t) + x(t) = 0 \tag{173}$$

has a general solution

$$x(t) = B\cos t + C\sin t$$

with arbitrary constants B, C. Can be checked by direct verification (note that $\ddot{x} = -x$). The values of B, C are determined by *initial conditions* $x(0)$ and $\dot{x}(0)$. Alternatively, one can combine sin and cos:

$$x(t) = A\cos(t + \phi)$$

with two constants $A = \sqrt{B^2 + C^2}$, and ϕ.

The equation

$$\ddot{x}(t) + \omega^2 x(t) = 0 \tag{174}$$

is reduced to the above by replacing t with ωt. Thus,

$$x(t) = B\cos(\omega t) + C\sin(\omega t) \tag{175}$$

with

$$B = x(0)\,,\ C = \dot{x}(0)/\omega \tag{176}$$

Or,

$$x(t) = A\cos\left(\omega t + \phi\right)\ ,\ A = \sqrt{B^2 + C^2} \tag{177}$$

with A known as the *amplitude* and ϕ the initial *phase*.

B. Spring pendulum

Hook's law:

$$F = -kx \tag{178}$$

and 2nd Newton's law

$$F = m\ddot{x} \tag{179}$$

give eq. (174) with

$$\omega = \sqrt{\frac{k}{m}} \tag{180}$$

(in radians per second). The oscillation frequency

$$f = \omega/2\pi \tag{181}$$

with units $1/s$, or Hz. Period of oscillations

$$T = \frac{1}{f} = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{m}{k}} \tag{182}$$

1. *Energy*

Kinetic:

$$K(t) = \frac{1}{2}m\left[\dot{x}(t)\right]^2$$

Potential:

$$U(t) = \frac{1}{2}kx(t)^2$$

Total:

$$E = K + U = const$$

C. Simple pendulum

Restoring force:

$$F = -mg\sin\theta \simeq -mg\theta$$

Tangential acceleration:

$$a = L\ddot{\theta}$$

Thus

$$\ddot{\theta} + \frac{g}{L}\theta = 0 \tag{183}$$

exactly like eq. (174). Thus, the same solution with $x \rightarrow \theta$ and

$$\omega^2 = \frac{g}{L} \tag{184}$$

or

$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{L}{g}} \tag{185}$$

D. Physical pendulum

Rotational 2d law:

$$I\alpha \equiv I\ddot{\theta} = \tau$$

If l - distance from CM:

$$\tau = -mgl\sin\theta \approx -mgl\theta$$

for small amplitudes.

Thus,

$$\ddot{\theta} + \theta\frac{mgl}{I} = 0 \tag{186}$$

which is the same differential equation as before with $x(t) \rightarrow \theta(t)$. Thus, the same trigonometric solution with

$$\omega^2 = \frac{mgl}{I}\,,\ T = 2\pi\sqrt{\frac{I}{mgl}} \tag{187}$$

Example: uniform rod of length L, pivoted at a distance l from the center.

Solution: from the rotational inertia of a rod about the CM, $I_0 = \frac{1}{12}ML^2$ and the parallel axis theorem

$$I = I_0 + Ml^2 = M\left(L^2/12 + l^2\right)$$

Thus,

$$T = 2\pi\sqrt{\frac{I}{Mgl}} = 2\pi\sqrt{\frac{L^2/12 + l^2}{gl}}$$

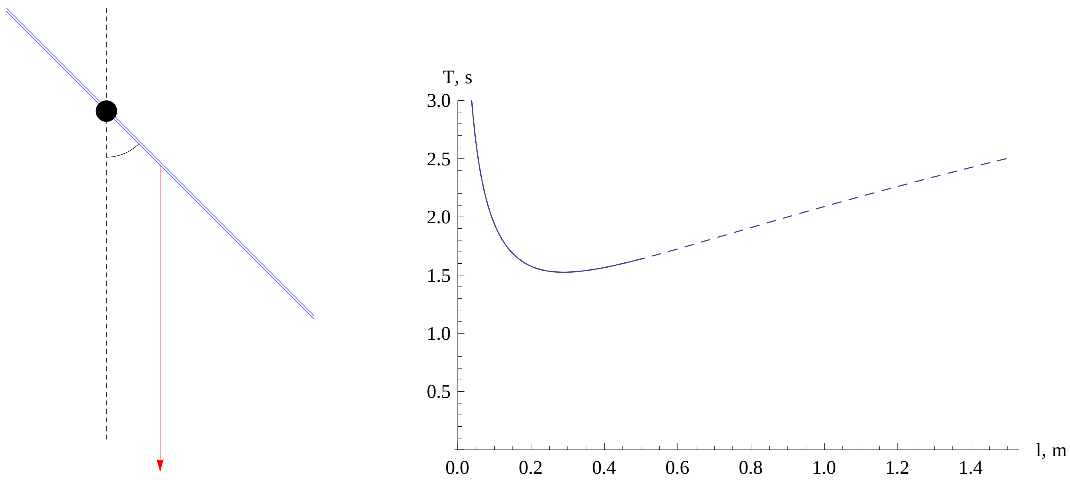


FIG. 36: Physical pendulum of mass M and length L making angle θ with vertical. Forces: $M\vec{g}$ (red), with torque $\tau = -Mgl\sin\theta$. Right: period of small oscillations T for $L = 1\,m$ as a function of the off-center distance l, with a minimum at $l \approx 29\,cm$; dashed line corresponds to the pivot outside of the rod (on a massless extension).

E. Torsional pendulum

Consider

$$\tau = -\kappa\theta$$

which is a torsional "Hook's law". Then, from rotational 2d law:

$$I\alpha \equiv I\ddot{\theta} = \tau$$

and

$$\ddot{\theta} + \theta\frac{\kappa}{I} = 0 \tag{188}$$

Thus,

$$\omega^2 = \frac{\kappa}{I}\,,\ T = 2\pi\sqrt{\frac{I}{\kappa}} \tag{189}$$

F. Why are small oscillations so universal?

in class

G. Resonance

Add external driving to a spring pendulum:

$$F = -kx + F_0 \cos(\omega_d t)$$

Then

$$m\ddot{x} + kx = F_0 \cos(\omega_d t) \tag{190}$$

Look for a solution

$$x(t) = A \cos(\omega_d t)$$

where the amplitude A has to be found. Using

$$\ddot{x}(t) = -\omega_d^2 x(t)$$

one obtains

$$A\left(-m\omega_d^2 + k\right) = F_0$$

or, with $\omega_0 = \sqrt{k/m}$, the natural frequency

$$|A| = \frac{F_0/m}{|\omega_0^2 - \omega_d^2|} \tag{191}$$

Note INFINITY when $\omega_d = \omega_0$. This is the resonance!

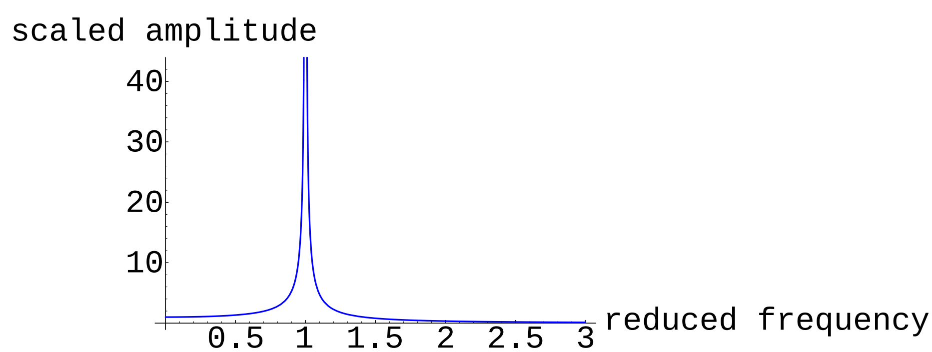


FIG. 37: Resonance. When the driving frequency ω_d is close to the natural frequency $\omega = \sqrt{k/m}$ there is an enormous increase in the amplitude.